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Technical Report

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E. J. Kelly

Finite-Sum Expressions for Signal Detection Probabilities

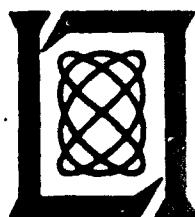
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FINITE-SUM EXPRESSIONS FOR SIGNAL DETECTION PROBABILITIES

E. J. KELLY

Group 41

TECHNICAL REPORT 566

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ABSTRACT

A unified approach is applied to the derivation of a number of formulas for the probability of signal detection and the probability of false alarm. The context is incoherent integration, with fluctuating signal-to-noise ratios and/or fluctuating thresholds. The standard results are obtained and extended to more general fluctuation models. A fundamental duality is established between fluctuating signals and fluctuating thresholds and used to simplify the derivations. Also included is an expression of the cumulative F-distribution as a finite sum of Marcum Q-functions.

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I. INTRODUCTION

In a wide variety of signal detection and binary decision problems, the evaluation of performance reduces to the computation of a number, P , defined as the probability that one random variable, u , exceeds another, called y . In symbols,

$$P = \text{Prob } \{u > y\} ,$$

and u represents a processor output which is being tested for signal presence, while y is a threshold, which may be a constant. In all cases considered here, u and y are independent, and u is the result of incoherent summation of N complex samples each containing signal and noise components. These quantities, z_n , represent I and Q output samples of some coherent processor and they are modelled as sums of signal and noise terms:

$$z_n = s_n + w_n .$$

The noise components are Gaussian, independent and with zero-mean, all sharing the same variance:

$$E|w_n|^2 = 2\sigma^2 ,$$

and all having the circular property

$$Ew_n^2 = 0.$$

Various models will be used for the signal components, corresponding to (and extending) most of the "fluctuating target" models of the radar literature.⁽¹⁾ The random variable u is the normalized sum

$$u = \frac{1}{2\sigma^2} \sum_{n=1}^N |z_n|^2 , \quad (1-1)$$

hence its probability distribution function (pdf), conditioned on the signal sequence, $\{s_n\}$, is

$$f_N(u, a) = e^{-u-a} (u/a)^{\frac{N-1}{2}} I_{N-1}(2\sqrt{au}). \quad (1-2)$$

In this formula,

$$a = \frac{1}{2\sigma^2} \sum_{n=1}^N |s_n|^2, \quad (1-3)$$

and I_n is the Bessel function of imaginary argument. Since the signal components enter this conditioned pdf only through the sum a , the "random signal" models will be represented by postulated pdf's for a .

In a similar way, "fluctuating thresholds," such as those arising in CFAR problems, are represented by putting $y=cx$, where c is a scalar multiplier, and x is a random variable. Various pdf's are postulated for x , including the pdf $f_L(x, d)$, with fixed or fluctuating parameter, d . Ordinary "linear CFAR" is represented by this model with $d=0$.

When a and y are constant, P is the standard Marcum Q-function,^(2,3) which must be computed from an infinite series, used together with a bound on truncation error. An efficient algorithm has been described by Shnidman,⁽⁴⁾ which involves recursive computation of the terms, along with several refinements which improve computational efficiency. The standard series, (Fehlner's formula, see Section 6) was used by Mitchell and Walker⁽⁵⁾ as the basis of a systematic derivation of expressions for some of the cases presented here. Because they start with an infinite series for the Q-function and obtain their results by averaging over various random models for a and/or y , all the expressions obtained are in infinite series form. Although all these series solutions lend themselves readily to recursive computation, it

happens that in most of the cases (except the original one, where both a and y are constant), finite-sum expressions exist and many of these can be found in the literature. ⁽¹⁾

In this study, we start with an integral representation for the Q -function, instead of an infinite series. Then, in analogy to the method of Mitchell and Walker, we derive the desired results for random a and/or y by averaging this expression under the sign of integration. Then, by simple changes of the variable of integration, the integral representation is itself used to obtain the desired finite-sum expressions. In all but two cases, the terms of these sums are elementary functions, easily computed recursively. In the two exceptions, the terms of the sums are themselves Q -functions. One of these, actually representing the cumulative F -distribution, is the source of a series of generalizations of the standard fluctuation models.

When $a=0$ in formula (1-2), the resulting pdf will be called $g_N(u)$:

$$g_N(u) = f_N(u, 0) = \frac{u^{N-1}}{(N-1)!} e^{-u} \quad (1-4)$$

This function, the chi-squared pdf with $2N$ degrees of freedom, is used to model signal and threshold fluctuations for the standard formulas derived in Section 4. Only integral values of N are used, since non-integral values (as in the Weinstock cases⁽⁶⁾) do not lead to detection formulas of finite form. Fluctuating signals are modelled by postulating that the pdf of a is

$$f_0(a) = \frac{1}{b} g_M\left(\frac{a}{b}\right) \quad (1-5)$$

where M and b are fluctuation parameters. In this model, the mean value of a is

$$\bar{a} = Mb, \quad (1-6)$$

and according to (1-3),

$$\bar{a} = \frac{1}{2\sigma^2} \sum_{n=1}^N |s_n|^2 . \quad (1-7)$$

We denote the average signal-to-noise ratio for a single sample by "SNR," so that

$$\text{SNR} = \frac{a}{N}$$

for a fixed signal, and

$$\text{SNR} = \frac{\bar{a}}{N} = \frac{M}{N} b \quad (1-8)$$

for a signal fluctuating according to (1-5). The standard "Swerling cases" correspond to M-values, as follows:⁽⁷⁾

<u>Swerling Case</u>	<u>M Value</u>
1	1
2	N
3	2
4	2N

When (1-4) is used to describe a fluctuating threshold, we write

$$y = cx,$$

and use

$$g_L(x)$$

for the pdf of x. The pdf of y is then

$$f_o(y) = \frac{1}{c} g_L\left(\frac{y}{c}\right) . \quad (1-9)$$

This model represents linear CFAR, and the variable x is the result of incoherent summation of L complex samples containing only noise of the same variance as that of the signal samples. The terms of this sum are normalized, just as in (1-1), by the true noise variance, σ^2 , but these normalizing factors cancel when the probability that u exceeds y is computed. If the noise samples are known to have a different variance than the signal samples, the variance of the signal samples is used for normalization and the variance ratio is absorbed into the constant, c . Note that x is the sum of the noise samples, not the average, which affects only the significance of c .

The fluctuation models are generalized in Section 5. First, the pdf,

$$f_o(a) = \frac{1}{b} f_M\left(\frac{a}{b}, d\right), \quad (1-10)$$

is used to describe signal fluctuations. The expected value of a in this case is $\bar{a} = (M+d)b$, hence

$$\text{SNR} = \frac{M}{N} b(1 + \frac{d}{M}) \quad . \quad (1-11)$$

This case is of rather academic interest, and the dual problem with fixed signal and threshold $y=cx$, is more interesting. The pdf of x is taken to be

$$f_o(x) = f_L(x, d),$$

which models linear CFAR, with L samples in the threshold, but with signal components (unwanted in the CFAR application) included in these samples. Only the sum of these signal powers affects the pdf, and the average signal-to-noise ratio, per threshold sample, will be d/L .

In further generalizations, a is allowed to fluctuate, using the same models as in Section 4 and also (separately and together) the new threshold signal components are randomized using the pdf

$$f_o(d) = \frac{1}{h} g_K\left(\frac{d}{h}\right), \quad (1-12)$$

for the parameter d . The average SNR, per threshold sample, in this last case is Kh/L .

A key feature of the basic approach is the duality between fluctuating signals and fluctuating thresholds. In Section 3 a useful artifice is introduced which allows us to obtain the detection probability for a fixed signal and threshold fluctuating according to a pdf, $f_o(y)$, from the solution of the dual problem, where y is constant and a fluctuates with the same pdf, $f_o(a)$.

The results are collected without proof in Section 2, and the basic method is developed in Section 3. The standard formulas are obtained in Section 4, and some extensions of these results are given in Section 5.

Finally, in Section 6, some alternative finite forms are obtained and for one common case a derivation of the finite form directly from the equivalent infinite series is given.

II. COLLECTED RESULTS

The results of this study are collected here without proofs, all of which will be found in the next two sections. In all cases, the assumptions and conditions are spelled out in detail, but detection and false alarm probabilities are called simply P_D and P_{FA} , since the number and variety of parameters become too great to list as arguments in some of the cases. The following conventions are used for the most common parameters:

<u>Parameter</u>	<u>Significance</u>
N	Number of "signal-plus-noise" samples integrated for detection.
L	Number of "noise-only" samples integrated to establish a CFAR threshold.
M	Signal fluctuation parameter, corresponding to chi-squared distribution with $2M$ degrees of freedom.

K	Fluctuation parameter for signal components in threshold samples, again chi-squared, with 2K degrees of freedom.
a	Signal parameter for fixed signals
b	Signal parameter for fluctuating signals
c	CFAR threshold multiplier
d	Parameter for fluctuating signal components in threshold samples

Three functions enter these formulas. These are defined here and discussed elsewhere.

$$(1) \quad P_N(y, a) = \int_y^{\infty} f_N(u, a) du , \quad (2-1)$$

where f_N is given by (1.2). $P_N(y, a)$ is the probability of detection for fixed, normalized total signal-to-noise ratio, a. It is essentially the Marcum Q-function for incoherent integration of N samples.

$$(2) \quad S_N(y) = P_N(y, 0) = e^{-y} \sum_{m=0}^{N-1} \frac{y^m}{m!} . \quad (2-2)$$

$S_N(y)$ is the false alarm probability corresponding to $P_N(y, a)$. It is computed recursively:

$$\left. \begin{array}{l} S_{N+1}(y) = S_N(y) + T_N(y) \\ T_N(y) = \frac{y}{N} T_{N-1}(y) \end{array} \right\} \quad N > 1 \quad (2-3)$$

$$S_1(y) = T_0(y) = e^{-y}$$

$$(3) \quad R_N(y, M) = \sum_{n=0}^{N-1} \binom{M-1+n}{n} y^n. \quad (2-4)$$

These functions are also computed recursively:

$$\left. \begin{array}{l} R_{N+1}(y, M) = R_N(y, M) + v_N(y, M) \\ v_N(y, M) = (1 + \frac{M-1}{N})y v_{N-1}(y, M) \end{array} \right\} \quad N > 1 \quad (2-5)$$

$$R_1(y, M) = v_0(y, M) = 1$$

The probability density functions used for the fluctuation models are

$$g_N(x) = \frac{x^{N-1}}{(N-1)!} e^{-x}$$

and

$$f_N(x, \alpha) = e^{-x-\alpha} \frac{x^{\frac{N-1}{2}}}{\alpha^{\frac{N-1}{2}}} I_{N-1}(2\sqrt{\alpha x})$$

A. Non-Fluctuating Signal

In these formulas the signal components of the random variable being tested for signal presence are constants. The average SNR per sample in all these cases is

$$\text{SNR} = a/N.$$

(1) Fixed Threshold, y

$$P_D = P_N(y, a) \quad (2-6)$$

$$P_{FA} = S_N(y) \quad (2-7)$$

(2) Fluctuating threshold, $y=cx$, pdf of x is

$$f_o(x) = g_L(x)$$

$$P_D = 1 - \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell S_{\ell+1} \left(\frac{a}{1+c} \right) \quad (2-8)$$

$$P_{FA} = \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell \quad (2-9)$$

(3) Fluctuating Threshold, $y=cx$, pdf of x is

$$f_o(x) = f_L(x, d)$$

$$P_D = \frac{c^N}{(1+c)^{N+L-1}} \left\{ \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c} \right)^{\ell+1} P_{\ell+1} \left(\frac{cd}{1+c}, \frac{a}{1+c} \right) \right. \\ \left. + \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell [1 - P_{\ell+1} \left(\frac{a}{1+c}, \frac{cd}{1+c} \right)] \right\} \quad (2-10)$$

$$P_{FA} = \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c} \right)^{\ell+1} S_{\ell+1} \left(\frac{cd}{1+c} \right) \quad (2-11)$$

(4) Fluctuating Threshold, $y=cx$, conditional pdf of x is

$$f_o(x|d) = f_L(x,d) ; \text{ pdf of } d \text{ is}$$

$$f_o(d) = \frac{1}{h} g_K\left(\frac{d}{h}\right).$$

Condition: $K > L$

$$P_D = 1 - \frac{c^N (1+h)^{N+L-K}}{(1+c+ch)^{N+L-1}} \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch}\right)^\ell x$$

$$x \sum_{n=0}^{L-1+\ell} \binom{N+L-1+\ell}{N+n} (c+ch)^n s_{n+1} \left(\frac{a}{1+c+ch}\right)$$
(2-12)

$$P_{FA} = \frac{(1+h)^{L-K}}{(1+c+ch)^{N+L-1}} \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch}\right)^\ell x$$

$$x \sum_{n=0}^{N-1} \binom{N+L-1+\ell}{N+n} (c+ch)^n$$
(2-13)

B. Fluctuating Signal, pdf of a is

$$f_o(a) = \frac{1}{b} g_M\left(\frac{a}{b}\right) .$$

$$\text{Average SNR per sample: } \text{SNR} = \frac{Mb}{N} .$$

(1) Fixed Threshold, y . P_{FA} given by (2-7)

M>N:

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell s_{N+\ell} \left(\frac{y}{1+b}\right) \quad (2-14)$$

M<N:

$$P_D = (-1/b)^M \left(\frac{1+b}{b}\right)^{N-M} \left\{ \sum_{\ell=1}^M \binom{N-1-\ell}{N-M-1} (-b)^\ell s_\ell \left(\frac{y}{1+b}\right) \right. \\ \left. + \sum_{\ell=1}^{N-M} \binom{N-1-\ell}{M-1} \left(\frac{b}{1+b}\right)^\ell s_\ell(y) \right\} \quad (2-15)$$

Special cases:

M=N:

$$P_D = s_N \left(\frac{y}{1+b}\right) \quad (2-16)$$

M=2N:

$$P_D = \frac{1}{(1+b)^N} \sum_{\ell=0}^N \binom{N}{\ell} b^\ell s_{N+\ell} \left(\frac{y}{1+b}\right) \quad (2-17)$$

M=1 (N>2):

$$P_D = s_{N-1}(y) + \left(\frac{1+b}{b}\right)^{N-1} e^{-\frac{y}{1+b}} \left[1 - s_{N-1} \left(\frac{by}{1+b}\right) \right] \quad (2-18)$$

M=2 (N>3):

$$P_D = s_{N-2}(y) + \left(\frac{1+b}{b}\right)^{N-2} e^{-\frac{y}{1+b}} \left\{ \left(1 - \frac{N-2}{b}\right) \left[1 - s_{N-2} \left(\frac{by}{1+b}\right) \right] \right\}$$

$$+ \frac{y}{1+b} [1 - s_{N-3} \left(\frac{by}{1+b} \right)] \} \quad (2-19)$$

(2) Fluctuating threshold, $y=cx$, pdf of x is

$$f_o(x) = g_L(x).$$

P_{FA} given by (2-9).

Any M :

$$P_D = 1 - \frac{c^N (1+c)^{M-N-L+1}}{(1+b+c)^M} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell R_{\ell+1} \left(\frac{b}{1+b+c}, M \right) \quad (2-20)$$

$M > N$:

$$P_D = \frac{(1+b)^{N+L-M}}{(1+b+c)^L} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell R_{N+\ell} \left(\frac{c}{1+b+c}, L \right) \quad (2-21)$$

Special cases:

$M=N$:

$$P_D = \left(\frac{1+b}{1+b+c} \right)^L R_N \left(\frac{c}{1+b+c}, L \right) \quad (2-22)$$

$M=2N$:

$$P_D = \frac{(1+b)^{L-N}}{(1+b+c)^L} \sum_{\ell=0}^N \binom{N}{\ell} b^\ell R_{N+\ell} \left(\frac{c}{1+b+c}, L \right) \quad (2-23)$$

M=1:

$$P_D = \frac{(1+b)^{N+L-1}}{b^{N-1}(1+b+c)^L} + \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell \left\{ 1 - \left(\frac{b}{1+b+c} \right)^{\ell+1-N} \right\} \quad (2-24)$$

M=2:

$$P_D = \frac{(1+b)^{N+L-1}}{b^{N-2}(1+b+c)^{L+1}} \left\{ 1 - (N-2) \frac{1+c}{b} + (N+L-1) \frac{c}{1+b} \right\}$$

$$+ \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell \left\{ 1 - [1 + (\ell+2-N) \frac{1+c}{b}] \left(\frac{b}{1+b+c} \right)^{\ell+2-N} \right\} \quad (2-25)$$

(3) Fluctuating threshold, $y=cx$, pdf of x is

$$f_o(x) = f_L(x, d).$$

P_{FA} given by (2-11)

M>N:

$$P_D = \frac{c^{N-1}(1+b)^{N+L-M}}{(1+b+c)^{N+L-1}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} \left(\frac{bc}{1+b+c} \right)^\ell x$$

$$x \sum_{n=0}^{N+\ell-1} \binom{N+L+\ell-1}{L+n} \left(\frac{1+b}{c} \right)^n S_{n+1} \left(\frac{cd}{1+b+c} \right) \quad (2-26)$$

Special cases:

M=N:

$$P_D = \frac{c^{N-1}(1+b)^L}{(1+b+c)^{N+L-1}} \sum_{n=0}^{N-1} \binom{N+L-1}{L+n} \left(\frac{1+b}{c}\right)^n s_{n+1} \left(\frac{cd}{1+b+c}\right) \quad (2-27)$$

M=2N:

$$P_D = \frac{c^{N-1}(1+b)^{L-N}}{(1+b+c)^{N+L-1}} \sum_{\ell=0}^N \binom{N}{\ell} \left(\frac{bc}{1+b+c}\right)^\ell x \\ \times \sum_{n=0}^{N+\ell-1} \binom{N+L+\ell-1}{L+n} \left(\frac{1+b}{c}\right)^n s_{n+1} \left(\frac{cd}{1+b+c}\right) \quad (2-28)$$

M=1:

$$P_D = \frac{(1+b)^{N+L-1}}{b^{N-1}(1+b+c)^L} e^{-\frac{cd}{1+c+d}} + \frac{c^{N-1}}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} x \\ \times \left\{ \left(\frac{1}{c}\right)^\ell s_{\ell+1} \left(\frac{cd}{1+c}\right) - e^{-\frac{cd}{1+c+d}} \left(\frac{1+b+c}{bc}\right)^\ell s_{\ell+1} \left[\frac{bcd}{(1+c)(1+b+c)} \right] \right\} \quad (2-29)$$

(4) Fluctuating threshold, $y=cx$, conditional pdf of x is

$$f_o(x|d) = f_L(x, d), \text{ pdf of } d \text{ is}$$

$$f_o(d) = \frac{1}{h} g_K \left(\frac{d}{h}\right)$$

Any $M, K > L$:

$$P_D = 1 - \frac{c^N (1+h)^{N+L-K} (1+c+ch)^{M-N-L+1}}{(1+b+c+ch)^M} \times$$

$$\times \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch}\right)^\ell \sum_{n=0}^{L+\ell-1} \binom{N+L+\ell-1}{N+n} (c+ch)^n R_{n+1} \left(\frac{b}{1+b+c+ch}, M\right) \quad (2-30)$$

P_{FA} given by (2-13)

Any K , $M > N$:

$$P_D = \frac{c^{N-1} (1+b)^{N+L-M} (1+b+c)^{K-N-L+1}}{(1+b+c+ch)^K} \times$$

$$\times \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} \left(\frac{bc}{1+b+c}\right)^\ell \sum_{n=0}^{N+\ell-1} \binom{N+L+\ell-1}{L+n} \left(\frac{1+b}{c}\right)^n R_{n+1} \left(\frac{ch}{1+b+c+ch}, K\right) \quad (2-31)$$

$$P_{FA} = \frac{c^{N-1} (1+c)^{K-N-L+1}}{(1+c+ch)^K} \sum_{n=0}^{N-1} \binom{N+L-1}{L+n} \left(\frac{1}{c}\right)^n R_{n+1} \left(\frac{ch}{1+c+ch}, K\right) \quad (2-32)$$

Special Cases:

$K=L$, any M :

$$P_D = 1 - \frac{c^N (1+h)^N (1+c+ch)^{M-N-L+1}}{(1+b+c+ch)^M} \times$$

$$\times \sum_{n=0}^{L-1} \binom{N+L-1}{N+n} (c+ch)^n R_{n+1} \left(\frac{b}{1+b+c+ch}, M\right) \quad (2-33)$$

$$P_{FA} = \frac{1}{(1+c+ch)^{N+L-1}} \sum_{n=0}^{N-1} \binom{N+L-1}{n} (c+ch)^n \quad (2-34)$$

$M=N$, any K :

$$P_D = \frac{c^{N-1} (1+b)^L (1+b+c)^{K-N-L+1}}{(1+b+c+ch)^K} \sum_{n=0}^{N-1} \binom{N+L-1}{L+n} \left(\frac{1+b}{c}\right)^n R_{n+1} \left(\frac{ch}{1+b+c+ch}\right), \quad K \quad (2-35)$$

P_{FA} given by (2-32)

C. Fluctuating signal, pdf of a is

$$f_o(a) = \frac{1}{b} f_M\left(\frac{a}{b}, d\right) .$$

Average SNR per sample: $SNR = \frac{b}{N} (1+d)$

(1) Fixed threshold, y . P_{FA} given by (2-7)

$M > N$:

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell P_{N+\ell} \left(\frac{y}{1+b}, \frac{bd}{1+b}\right) \quad (2-36)$$

It is often required to evaluate detection performance when the false alarm probability is fixed, which involves the inversion of the formula for P_{FA} in order to determine the threshold. The Newton-Raphson iterative method⁽⁸⁾ is useful for this purpose, and we discuss its application to the two most common false alarm formulas, (2-7) and (2-9). The other cases listed above involve CFAR problems with signal components in the threshold samples, but these components are usually not anticipated, and the threshold will have

been found from (2-9). Formulas like (2-11) and (2-13) then show the effect of the signal components on P_{FA} after the threshold has been fixed.

Beginning with (2-7), we want to solve the equation,

$$\phi(y) \equiv S_N(y) - P_{FA} = 0,$$

for y , where P_{FA} is the assigned probability of false alarm. The Newton-Raphson iteration is

$$y_{n+1} = G(y_n),$$

where

$$G(y) = y - \frac{\phi(y)}{\phi'(y)},$$

and y_0 is a suitable initial value. A good plan is to start the iteration at a point where the derivative, $\phi'(y)$, is large, to avoid wild oscillation of the sequence, y_n . This can be assured by choosing y_0 to be the solution of

$$\phi''(y_0) = 0,$$

which will be unique in the applications made here.

From (2-2) we have

$$\phi'(y) = S'_N(y) = -e^{-y} \sum_{m=0}^{N-1} \frac{y^m}{m!} + e^{-y} \sum_{m=0}^{N-2} \frac{y^m}{m!}$$

$$= -e^{-y} \frac{y^{N-1}}{(N-1)!}$$

In (2-3) we have written, in effect,

$$S_N(y) = \sum_{n=0}^{N-1} T_n(y),$$

where

$$T_n(y) = e^{-y} \frac{y^n}{n!},$$

hence

$$S'_N(y) = -T_{N-1}(y).$$

As $S_N(y)$ is computed recursively, from (2-3), for use in the iteration, the final term, $T_{N-1}(y)$, can be recovered for the evaluation of $\phi'(y)$. Then

$$G(y) = y + \frac{S_N(y) - P_{FA}}{T_{N-1}(y)},$$

and iteration is continued until the correction term, $G(y)-y$, reaches some preassigned small value. We also find

$$\phi''(y) = \frac{e^{-y}}{(N-1)!} [y^{N-1} - (N-1)y^{N-2}],$$

and so

$$y_0 = N-1$$

is the desired starting value.

A very similar procedure can be used for (2-9) with

$$\phi(y) = \frac{1}{(1+y)^{N+L-1}} \sum_{k=0}^{N-1} \binom{N+L-1}{k} y^k - p_{FA}.$$

(we have retained the symbol y , instead of the c of (2-9)).

This time,

$$\begin{aligned} \phi'(y) &= - \frac{N+L-1}{(1+y)^{N+L}} \sum_{k=0}^{N-1} \binom{N+L-1}{k} y^k \\ &+ \frac{1}{(1+y)^{N+L-1}} \sum_{k=1}^{N-1} \frac{(N+L-1)!}{(N+L-k-1)!(k-1)!} y^{k-1} \\ &= - \frac{N+L-1}{(1+y)^{N+L}} \left\{ \sum_{k=0}^{N-1} \binom{N+L-1}{k} y^k - \sum_{k=1}^{N-1} \binom{N+L-2}{k-1} (y^{k-1} + y^k) \right\} \end{aligned}$$

Using the reduction formula

$$\binom{N+L-1}{k} = \binom{N+L-2}{k} + \binom{N+L-2}{k-1},$$

the quantity in curly brackets becomes

$$\sum_{k=0}^{N-1} \binom{N+L-2}{k} y^k - \sum_{k=1}^{N-1} \binom{N+L-2}{k-1} y^{k-1} = \binom{N+L-2}{N-1} y^{N-1} .$$

Therefore

$$\begin{aligned}\phi'(y) &= - \frac{N+L-1}{(1+y)^{N+L}} \binom{N+L-2}{N-1} y^{N-1} \\ &= - \frac{L}{1+y} \cdot \frac{1}{(1+y)^{N+L-1}} \binom{N+L-1}{N-1} y^{N-1} ,\end{aligned}$$

and if we define

$$W_N(y) \equiv \frac{1}{(1+y)^{N+L-1}} \sum_{k=0}^{N-1} \binom{N+L-1}{k} y^k$$

$$\equiv \sum_{k=0}^{N-1} Q_k(y),$$

then

$$Q_k(y) = \frac{1}{(1+y)^{N+L-1}} \binom{N+L-1}{k} y^k$$

and

$$\phi'(y) = - \frac{L}{1+y} Q_{N-1}(y).$$

Thus the iterative algorithm is based upon

$$G(y) = y + \frac{1+y}{L} \frac{W_N(y) - P_{FA}}{Q_{N-1}(y)} ,$$

and $Q_{N-1}(y)$ is recovered from the iterative computation of $W_N(y)$:

$$W_N(y) = W_{N-1}(y) + Q_{N-1}(y) ,$$

$$Q_L(y) = \left(\frac{N+L}{L} - 1\right)y Q_{L-1}(y) ,$$

$$W_1(y) = Q_0(y) = \frac{1}{(1+y)^{N+L-1}}$$

Finally, we compute

$$\phi''(y) = -L \left(\frac{N+L-1}{N-1}\right) \frac{(N-1)y^{N-2}(1+y) - (N+L)y^{N-1}}{(1+y)^{N+L-1}}$$

and therefore

$$y_0 = \frac{N-1}{L+1}$$

provides a suitable initial value.

III. BASIC FORMULATION

The whole analysis here is based on Schlafli's integral representation⁽⁹⁾ for the Bessel functions. We use it in the form

$$I_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{t + \frac{x^2}{4t}} \frac{dt}{t^{n+1}}. \quad (3-1)$$

The contour of integration is a small circle, enclosing the origin in a positive sense. With this representation, the pdf of u (for fixed a),

$$f_N(u, a) = e^{-u-a} \left(\frac{u}{a}\right)^{\frac{N-1}{2}} I_{N-1}(2\sqrt{au}), \quad (3-2)$$

can be written

$$f_N(u, a) = e^{-u-a} u^{N-1} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{t + \frac{au}{t}} \frac{dt}{t^N}.$$

Replacing t by ut in the integral, we obtain

$$f_N(u, a) = e^{-u-a} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{ut + \frac{a}{t}} \frac{dt}{t^N}. \quad (3-3)$$

This representation is more convenient than the Fourier transform expression, which can be obtained from it by the change of variable $t=1-i\lambda$. We carry out this transformation, since we need the characteristic function of f_N later on. After substitution, we have

$$f_N(u, a) = e^{-a} \frac{1}{2\pi i} \int_C e^{-iu\lambda + \frac{a}{1-i\lambda}} \frac{(-id\lambda)^N}{(1-i\lambda)^N}$$

where C is a small circle enclosing the point $\lambda = -i$. This contour can be expanded, without changing the integral, until it runs along the real λ -axis from $+A$ to $-A$, and then back along an arc of radius A , in the lower half-plane, until it closes at $\lambda = +A$ again. Because of the factor $\exp(-iu\lambda)$ in the integrand, the integral along the arc vanishes in the limit $A \rightarrow \infty$, for any $N=1, 2, \dots$. We then reverse the path along the real axis to obtain

$$f_N(u, a) = e^{-a} \int_{-\infty}^{\infty} e^{-iu\lambda} \frac{e^{\frac{a}{1-i\lambda}}}{(1-i\lambda)^N} \frac{d\lambda}{2\pi} .$$

The inverse transform gives the characteristic function

$$\int_0^{\infty} e^{i\lambda u} f_N(u, a) du = e^{-a} \frac{e^{\frac{a}{1-i\lambda}}}{(1-i\lambda)^N} \equiv \phi_N(\lambda, a) . \quad (3-4)$$

Recalling the definition of a in equation (1-3), we can write $\phi_N(\lambda, a)$ as a product:

$$\phi_N(\lambda) = \prod_{n=1}^N \left\{ e^{-a_n} \frac{e^{\frac{a_n}{1-i\lambda}}}{1-i\lambda} \right\} ,$$

where $a_n = |s_n|^2/2\sigma^2$, which corresponds to the definition of the random variable, u , as a sum. Each factor here is the characteristic function of a non-central chi-squared random variable, with two degrees of freedom, and u is non-central chi-squared with $2N$ degrees of freedom.

From our basic integral representation for $f_N(u, a)$ we obtain the "no signal" special case,

$$f_N(u, 0) = e^{-u} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{ut} \frac{dt}{t^N} .$$

This set of pdf's will be used to model fluctuating thresholds and signal parameters later on, and we introduce the notation

$$g_N(u) \equiv f_N(u, 0) .$$

Evaluating the contour integral for $f_N(u, 0)$, we get

$$g_N(u) = \frac{1}{(N-1)!} u^{N-1} e^{-u} , \quad (3-5)$$

which is, of course, the chi-squared pdf, for $2N$ degrees of freedom. Obviously, $Eu=N$, if $g_N(u)$ is the pdf of u .

The "probability of detection" corresponding to the pdf $f_N(u, a)$ is denoted $P_N(y, a)$ and it is defined by the equation

$$P_N(y, a) \equiv \int_y^{\infty} f_N(u, a) du . \quad (3-6)$$

Our notation differs from the standard one⁽⁴⁾ in that we retain the same order of the variables as in the pdf, and the parameter a is the "total SNR," as defined in Section 1. The advantages of this choice seem worth the cost of departing from common usage. In terms of P_N , Marcum's: Q-function is

$$Q_N(\alpha, \beta) = P_N\left(\frac{\beta^2}{2}, \frac{\alpha^2}{2}\right) .$$

To obtain an integral representation for P_N , we substitute the representation (3-3) in definition (3-6) and reverse the order of integration:

$$P_N(y, a) = e^{-a} \frac{1}{2\pi i} \int_{|t|=\epsilon}^{\frac{a}{t}} e^{\frac{a}{t}} \int_y^{\infty} e^{-(1-t)u} du \frac{dt}{t^N} .$$

Since $\operatorname{Re}\{1-t\} > 0$ on the t -contour, the u -integral is uniformly convergent, thus validating the interchange, and we obtain

$$P_N(y, a) = e^{-y-a} \frac{1}{2\pi i} \int_{|t|=\epsilon}^{\frac{yt+a}{t}} e^{\frac{yt+a}{t}} \frac{dt}{t^N(1-t)} . \quad (3-7)$$

This is the fundamental representation for P_N , used throughout this study. A second representation is obtained by making the change of variable, $t+1/t$, in (3-7):

$$P_N(y, a) = e^{-y-a} \frac{1}{2\pi i} \int_{|t|=1/\epsilon}^{\frac{at+y}{t}} e^{\frac{at+y}{t}} \frac{t^{N-1} dt}{t-1} .$$

The contour now is a large circle, including the simple pole at $t=1$, as well as the essential singularity at the origin. The residue of the integral at the simple pole is $\exp(a+y)$, and hence we can write

$$P_N(y, a) = 1 - e^{-y-a} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{at + \frac{y}{t}} \frac{t^{N-1} dt}{1-t}, \quad (3-8)$$

where the contour is again a small circle enclosing the origin.

Both representations are useful and we note that the integral in (3-8) is just like the one in (3-7), except that the variables y and a are interchanged, and the factor t appears with a non-negative exponent. In other words, we have the evaluations

$$e^{-y-a} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{yt + \frac{a}{t}} \frac{t^{-n} dt}{1-t} = \begin{cases} P_n(y, a) & ; n>0 \\ 1-P_{1-n}(a, y) & ; n<0 \end{cases}$$

It proves to be exceedingly useful to define $P_N(y, a)$ for all integral N by means of representation (3-7), with the understanding that

$$P_{-N}(y, a) \equiv 1 - P_{N+1}(a, y). \quad (3-9)$$

This artifice allows us to exploit the duality between threshold and signal parameter in the following way. Suppose the signal parameter is randomized, according to some pdf, $f_o(a)$. With fixed threshold, y , the probability of detection will be

$$F_N(y) \equiv \int_0^{\infty} P_N(y, a) f_o(a) da, \quad (3-10)$$

Now suppose the right side is evaluated for positive and negative N , using (3-7) as the definition of $P_N(y, a)$. Then the dual problem, in which a is fixed and y is random (with the same pdf) has detection probability

$$\begin{aligned} \int_0^\infty P_N(y, a) f_0(y) dy &= \int_0^\infty [1 - P_{1-N}(a, y)] f_0(y) dy \\ &= 1 - F_{1-N}(a). \end{aligned} \quad (3-11)$$

This technique saves the needless duplication of calculations, which would otherwise use (3-7) for one set of problems, and (3-8), in a parallel way, with the dual problems.

The source of this duality is, of course, the basic pdf. The fact that $I_n(x) = I_{-n}(x)$ can be proved from (3-1) by making the substitution $t \rightarrow \frac{x}{4t}$ in the integral and hence from (3-2),

$$\begin{aligned} f_{N+1}(u, a) &= e^{-u-a} \left(\frac{u}{a}\right)^{\frac{N}{2}} I_N(2\sqrt{au}) \\ &= e^{-a-u} \left(\frac{a}{u}\right)^{-\frac{N}{2}} I_{-N}(2\sqrt{au}) . \end{aligned}$$

Thus we could define

$$f_{1-N}(a, u) = f_{N+1}(u, a), \quad (3-12)$$

and this definition is consistent with (3-6), extended to negative values of N . Integration of both sides of (3-12) over a yields

$$\int_b^\infty f_{1-N}(a, u) da = P_{1-N}(b, u) = 1 - P_N(u, b)$$

$$= \int_b^\infty f_{N+1}(u, a) da,$$

or

$$P_N(y, a) = 1 - \int_a^\infty f_{N+1}(y, a') da', \quad (3-13)$$

an interesting result which also follows directly from (3-3).

The probability of false alarm for fixed threshold will be denoted

$$S_N(y) = P_N(y, 0).$$

From (3-7) we have the integral representation

$$S_N(y) = e^{-y} \frac{1}{2\pi i} \int_{|t|=\epsilon}^\infty e^{yt} \frac{dt}{t^N(1-t)}, \quad (3-14)$$

which will often be used to evaluate contour integrals in this analysis. For $N > 1$, we have

$$S_N(y) = e^{-y} \sum_{m=0}^{\infty} \frac{1}{2\pi i} \int_{|t|=\epsilon}^\infty e^{yt} \frac{dt}{t^{N-m}},$$

after expanding $(1-t)^{-1}$, and then

$$S_N(y) = e^{-y} \sum_{m=0}^{N-1} \frac{y^m}{m!} . \quad (3-15)$$

The series terminates, since there is no pole at $t=0$ when $m>N$. Of course,

$$S_N(y) = \int_y^{\infty} g_N(u) du ,$$

which again yields (3-15) after substitution of (3-5) and repeated integration by parts.

From (3-14) we have

$$S_n(x) = 0 , \quad n < 0 . \quad (3-16)$$

This is completely consistent with (3-9), since

$$S_{-N}(x) = P_{-N}(x, 0) = 1 - P_{N+1}(0, x) ,$$

and

$$P_N(0, x) = \int_0^{\infty} f_N(u, x) du = 1 ,$$

by normalization.

We proceed now with the general problem of detection with a fixed threshold and random signal parameter. As above we let $f_o(a)$ be the pdf of a , and we define the characteristic function

$$\phi_0(\lambda) \equiv \int_0^\infty e^{i\lambda a} f_0(a) da . \quad (3-17)$$

We evaluate the detection probability by substituting the integral representation, (3-7) into Eq. (3-10):

$$F_N(y) = \int_0^\infty f_0(a) \left\{ e^{-y-a} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{yt + \frac{a}{t}} \frac{dt}{t^N(1-t)} \right\} da .$$

We want to reverse the order of integration, which will involve the evaluation of

$$\int_0^\infty e^{-(1 - \frac{1}{t})a} f_0(a) da .$$

For this integral to converge uniformly, it is necessary that $\operatorname{Re}(1-t) > 0$, which will be satisfied if we can arrange to have $|t| > 1$. This is accomplished by expanding the original contour, $|t| = \epsilon$, until it becomes a circle with radius larger than unity, and compensating for the effect of the pole (at $t=1$) now included. In other words we write

$$e^{-y-a} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{yt + \frac{a}{t}} \frac{dt}{t^N(1-t)}$$

$$= 1 + e^{-y-a} \frac{1}{2\pi i} \int_{|t|>1} e^{yt + \frac{a}{t}} \frac{dt}{t^N(1-t)} ,$$

and then reverse the order of integration. The result is

$$F_N(y) = 1 + e^{-y} \frac{1}{2\pi i} \int_{|t|>1} e^{yt} \left\{ \int_0^\infty e^{-\left(1-\frac{1}{t}\right)a} f_0(a) da \right\} \frac{dt}{t^N(1-t)} .$$

The integral over a yields $\phi_0[i(1 - \frac{1}{t})]$, with uniform convergence to justify the interchange.

Now

$$|\phi_0(\lambda)| < 1$$

whenever $\text{Im}(\lambda) > 0$, and this upper half-plane maps into the exterior (and boundary) of the circle,

$$|t - \frac{1}{2}| = \frac{1}{2} ,$$

in the t -plane. It will be true in the cases considered here that $\phi_0(\lambda)$ is analytic when $\text{Im } \lambda > -\epsilon$, for a suitable positive ϵ , so that $\phi_0[i(1 - \frac{1}{t})]$ will remain analytic for a small but finite distance inside the circle

$|t - \frac{1}{2}| = \frac{1}{2}$. This will allow the t -contour to be shrunk just inside the unit circle. Since $\phi_0(0) = 1$, the pole at $t=1$ is simple, and hence

$$F_N(y) = 1 + e^{-y} \frac{1}{2\pi i} \int_{|t|=1} e^{yt} \phi_o[i(1 - \frac{1}{t})] \frac{dt}{t^N(1-t)} ,$$

or

$$F_N(y) = e^{-y} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \phi_o[i(1 - \frac{1}{t})] \frac{dt}{t^N(1-t)} . \quad (3-18)$$

This is the basic expression which will be used throughout this report.

IV. DERIVATION OF THE STANDARD FORMULAS

We begin by randomizing the signal parameter, using a general even-order chi-squared pdf for a . In particular, we postulate

$$f_o(a) = \frac{1}{b} g_{\frac{1}{2}}\left(\frac{a}{b}\right) , \quad (4-1)$$

where $g_{\frac{1}{2}}(x)$ is the pdf defined by Eq. (3-5). The characteristic function is then

$$\phi_o(\lambda) = \int_0^\infty e^{i\lambda a} g_{\frac{1}{2}}\left(\frac{a}{b}\right) \frac{da}{b}$$

$$= \int_0^\infty e^{ib\lambda x} g_{\frac{1}{2}}(x) dx$$

$$= (1 - ib\lambda)^{-\frac{1}{2}} ,$$

a special case of (3-4), namely $\phi_M(b\lambda, 0)$. With this model,

$$Eu = Ea + N = Mb + N;$$

and if $M = kN$, the result is the same as assigning to each component signal parameter, a_n , the pdf

$$f(a_n) = \frac{1}{b} g_k\left(\frac{a_n}{b}\right),$$

with each component independent of the others. In this case, $Ea_n = kb$. The Swerling cases 1, 2, 3 and 4 correspond to the M -values 1, N , 2 and $2N$, respectively.

We note that

$$\delta_0[i(1 - \frac{1}{t})] = \left[\frac{t}{(1+b)t-b} \right]^M,$$

and substitute in Eq. (3-18). The resulting expression for the detection probability is

$$P_D = e^{-y} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \frac{t^{M-N} dt}{[(1+b)t-b]^{M-1} (1-t)} \quad (4-2)$$

A basic difference appears now, depending on whether $M > N$ or $M < N$. We suppose, first, that $M > N$, so the only pole in (4-2) is at $t = b/(1+b)$. We shrink the contour to a small circle enclosing this pole and then make the change of variable

$$t = \frac{s+b}{1+b} \quad . \quad (4-3)$$

In the s-plane the pole is at the origin and we also have

$$1 - t = \frac{1-s}{1+b} ,$$

so that

$$\frac{dt}{1-t} = \frac{ds}{1-s} .$$

This substitution yields

$$P_D = e^{-\frac{y}{1+b}} \frac{1}{2\pi i} \int_{|s|=\epsilon} e^{\frac{y}{1+b}s} \frac{(s+b)^{M-N}}{s^{M-1}} \frac{ds}{s(1-s)} .$$

Next, we expand the factor $(s+b)^{M-N}$ in a binomial series:

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{k=0}^{M-N} \binom{M-N}{k} b^k e^{-\frac{y}{1+b}} \frac{1}{2\pi i} \int_{|s|=\epsilon} e^{\frac{y}{1+b}s} \frac{ds}{s^{N+k}(1-s)} .$$

and evaluate the concour integral by means of (3-14). The result is

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{k=0}^{M-N} \binom{M-N}{k} b^k S_{N+k} \left(\frac{y}{1+b} \right) , \quad (4-4)$$

the desired expression.

The false alarm probability is, of course,

$$P_{FA} = S_N(y) \quad (4-5)$$

as it must be for any fixed-threshold model. Note that our pdf, $f_0(a)$, approaches $\delta(a)$, as $b \rightarrow 0$, and $\phi_0(\lambda) \rightarrow 1$. One can also put

$$b = a_0/M$$

and let $M \rightarrow \infty$. In this case $f_0(a)$ approaches $\delta(a-a_0)$, and $\phi_0(\lambda) \rightarrow \exp(ia_0\lambda)$, while P_D is transformed into a standard, infinite-series expression for $P_N(y, a_0)$ (this is Fehlner's series, which is discussed in Section 6).

For $M=N$ we have the familiar Swerling case 2 result

$$P_D = S_N\left(\frac{y}{1+b}\right) \quad (M=N) ,$$

so favored by radar analysts. When $M=2N$, we obtain the Swerling case 4 expression

$$P_D = \frac{1}{(1+b)^N} \sum_{k=0}^N \binom{N}{k} b^k S_{N+k}\left(\frac{y}{1+b}\right) \quad (M=2N) .$$

When $M < N$ we write (4-2) as

$$P_D = e^{-y} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \frac{dt}{[(1+b)t-b]^M t^{N-M} (1-t)} , \quad (4-6)$$

and proceed differently. The integrand now has two poles, and we write

$$P_D = P_1 + P_2 , \quad (4-7)$$

where the integrands of P_1 and P_2 are the same as P_D , but the contour for P_1 is a small circle enclosing the pole at $t=b/(1+b)$, while for P_2 it is the circle $|t|=\epsilon$.

The evaluation of P_1 is the same as the case just analyzed, which used transformation (4-3), up to the point where

$$P_1 = e^{-\frac{y}{1+b}} \frac{1}{2\pi i} \int_{|s|=\epsilon} e^{\frac{y}{1+b}s} \left(\frac{s+b}{1+b}\right)^{-(N-M)} \frac{ds}{s^M(1-s)} .$$

This time the expansion is an infinite series:

$$\left(\frac{s+b}{1+b}\right)^{-(N-M)} = \left(\frac{1+b}{b}\right)^{N-M} \sum_{\ell=0}^{\infty} \binom{N-M-1+\ell}{\ell} (-s/b)^\ell ,$$

which converges since $|s|=\epsilon$ and ϵ can be chosen smaller than b . When this series is substituted and the integration performed termwise, only the terms with $\ell \leq M$ contribute, since beyond that there are no poles. Therefore,

$$P_1 = \left(\frac{1+b}{b}\right)^{N-M} \sum_{\ell=0}^{M-1} \binom{N-M-1+\ell}{\ell} (-1/b)^\ell S_{M-\ell} \left(\frac{y}{1+b}\right) , \quad (4-8)$$

where (3-14) has again been employed.

P_2 can be handled in a similar way, without changing the variable of integration from that used in (4-6) but expanding

$$[(1+b)t-b]^{-M} = \left(-\frac{1}{b}\right)^M \sum_{\ell=0}^{\infty} \binom{M-1+\ell}{\ell} \left(\frac{1+b}{b}t\right)^\ell.$$

Again, since $|t|=\epsilon$, convergence can be assured and we get

$$P_2 = \left(-\frac{1}{b}\right)^M \sum_{\ell=0}^{\infty} \binom{M-1+\ell}{\ell} \left(\frac{1+b}{b}\right)^\ell e^{-y} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^y \frac{dt}{t^{N-M-\ell}(1-t)}.$$

This time the sum stops at $\ell=N-M-1$, and the integrals are evaluated using (3-14). Altogether, after reordering the sums to get an increasing sequence of S-functions,

$$\begin{aligned} P_D = P_1 + P_2 &= \left(-\frac{1}{b}\right)^M \left(\frac{1+b}{b}\right)^{N-M} \left\{ \sum_{\ell=1}^M \binom{N-1-\ell}{N-M-1} (-b)^\ell S_\ell \left(\frac{y}{1+b}\right) \right. \\ &\quad \left. + \sum_{\ell=1}^{N-M} \binom{N-1-\ell}{M-1} \left(\frac{b}{1+b}\right)^\ell S_\ell(y) \right\} \end{aligned} \quad (4-9)$$

The usual application of these formulas will be to Swerling cases 1 and 3, where $M=1$ and $M=2$ respectively. For these or any small values of M , another expression is more convenient. It is obtained from (4-6) by writing

$$(1+b)t-b = \frac{t-x}{1-x},$$

where

$$x \equiv \frac{b}{1+b} < 1,$$

so that

$$\begin{aligned} [(1+b)t-b]^{-M} &= (1-x)^M (t-x)^{-M} \\ &= \frac{(1-x)^M}{(M-1)!} \left(\frac{d}{dx}\right)^{M-1} \left(\frac{1}{t-x}\right). \end{aligned}$$

Then (4-6) becomes

$$P_D = \frac{(1-x)^M}{(M-1)!} \left(\frac{d}{dx}\right)^{M-1} e^{-y} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \frac{dt}{t^{N-M} (t-x)(1-t)} . \quad (4-10)$$

We use the partial fraction expansion

$$\frac{1}{(t-x)(1-t)} = \frac{1}{1-x} \left(\frac{1}{1-t} - \frac{1}{x-t} \right)$$

and put

$$p_D = \frac{(1-x)^M}{(M-1)!} \left(\frac{d}{dx} \right)^{M-1} \left\{ \frac{\psi(x)}{1-x} \right\}$$

where

$$\psi(x) = e^{-y} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \left(\frac{1}{1-t} - \frac{1}{x-t} \right) \frac{dt}{t^{N-M}}$$

$$= e^{-y} \frac{1}{2\pi i} \int_{|t|=\epsilon} e^{yt} \frac{dt}{t^{N-M}(1-t)}$$

$$- \frac{e^{-y}}{x^{N-M}} \frac{1}{2\pi i} \int_{|t|>1} e^{xyt} \frac{dt}{t^{N-M}(1-t)} .$$

Note that the original contour was just inside the unit circle, but including the pole at $t=x$. Finally, again using (3-14), we get

$$\psi(x) = S_{N-M}(y) + \frac{(x-1)y}{x^{N-M}} [1 - S_{N-M}(xy)] .$$

Since

$$\frac{(1-x)^M}{(M-1)!} \left(\frac{d}{dx} \right)^{M-1} \frac{1}{1-x} = 1,$$

the complete expression is

$$P_D = S_{N-1}(y) + \frac{(1-x)^M}{(M-1)!} \left(\frac{d}{dx} \right)^{M-1} \left\{ \frac{e^{(x-1)y}}{(1-x)^{N-M}} [1 - S_{N-M}(xy)] \right\} \quad (4-11)$$

where

$$x = \frac{b}{1+b}$$

The identity

$$\frac{d}{dx} e^{xy} [1 - S_m(xy)] = y e^{xy} [1 - S_{m-1}(xy)] ,$$

which is easily verified, makes the evaluation of the derivatives in (4-11) relatively straight-forward.

For Swerling cases 1 and 3 we have:

$$P_D(M=1) = S_{N-1}(y) + \left(\frac{1+b}{b} \right)^{N-1} e^{-\frac{y}{1+b}} [1 - S_{N-1}(\frac{by}{1+b})] , \quad (4-12)$$

and

$$P_D(M=2) = S_{N-2}(y) + \left(\frac{1+b}{b} \right)^{N-2} e^{-\frac{y}{1+b}} \left\{ \left(1 - \frac{N-2}{b} \right) [1 - S_{N-2}(\frac{by}{1+b})] \right. \\ \left. + \frac{y}{1+b} [1 - S_{N-3}(\frac{by}{1+b})] \right\} \quad (4-13)$$

By writing

$$s_{N-1} \left(\frac{by}{1+b} \right) = s_N \left(\frac{by}{1+b} \right) - e^{-\frac{by}{1+b}} \frac{1}{(N-1)!} \left(\frac{by}{1+b} \right)^{N-1},$$

and substituting into (4-12), it follows that this equation can also be written

$$P_D = s_N(y) + \left(\frac{1+b}{b} \right)^{N-1} e^{-\frac{y}{1+b}} [1 - s_N \left(\frac{by}{1+b} \right)]. \quad (4-12)$$

In this form, (4-12) is also valid for $N=1$, in which situation Swerling cases 1 and 2 coincide. By a similar manipulation, (4-13) can be written

$$P_D = s_{N-1}(y) + e^{-y} \frac{y^{N-1}}{(1+b)(N-2)!} + \left(\frac{1+b}{b} \right)^{N-2} e^{-\frac{y}{1+b}} \left(1 - \frac{N-2}{b} + \frac{y}{1+b} \right) x \\ x [1 - s_{N-1} \left(\frac{by}{1+b} \right)] \quad (4-13)$$

which happens also to be valid for $N=2$, where Swerling cases 2 and 3 coincide.

We turn now to the dual problem of fluctuating threshold, as in linear CFAR detection. The signal parameter, a , is fixed, and the threshold is written

$$y = cx,$$

where the pdf of x is taken to be $g_L(x)$. Thus x corresponds to the sum of L random variables, each with the exponential pdf

$$g_1(x) = e^{-x} .$$

In other words, x is the result of incoherent summation of L complex samples of noise alone, the noise samples having the same variance as the noise components of the samples, z_n , which contribute to u . We can call these noise samples x_ℓ , and then

$$y = c \sum_{\ell=1}^L x_\ell = cL \cdot \frac{x_1 + \dots + x_L}{L} .$$

The average of the x_ℓ is the noise level estimate in CFAR problems, and the corresponding threshold multiplier is cL . With this model, the pdf of the threshold variable, y , is

$$f_0(y) = \frac{1}{c} g_L\left(\frac{y}{c}\right) , \quad (4-14)$$

and the required detection probability is

$$P_D = \int_0^\infty P_N(y, a) f_0(y) dy. \quad (4-15)$$

According to Eq. (3-11), this is obtainable from the solution of the dual problem:

$$P_D = 1 - \int_0^\infty P_{1-N}(a, y) f_0(y) dy$$

$$= 1 - \int_0^\infty P_{1-N}(a, y) g_L\left(\frac{y}{c}\right) \frac{dy}{c} .$$

This, of course, is the problem we have just solved and we have only to make the changes of variable

$$\begin{aligned} N+1-N \\ M+L \\ y+a \\ b+c \end{aligned}$$

in, for example, Eq. (4-2). We note immediately that the case previously called "M<N" cannot occur, since the old exponent $M-N$ becomes $N+L-1$ in the dual problem. Thus (4-4) provides the desired solution:

$$P_D = 1 - \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} c^\ell s_{\ell+1-N}\left(\frac{a}{1+c}\right) .$$

Since $s_m(x)$ vanishes unless $m>0$, only the terms in this sum having $\ell \geq N$ survive, and our result can be written in the form

$$P_D = 1 - \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell s_{\ell+1}\left(\frac{a}{1+c}\right) . \quad (4-16)$$

It may be verified that this result is also obtained by a direct calculation, starting from (4-15) and using representation (3-8) for P_N . This calculation closely parallels the derivations of Eqs (3-18) and (4-2).

The false alarm probability is obtained by putting $a=0$ in (4-16) and noting that $S_m(0)=1$:

$$P_{FA} = \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell , \quad (4-17)$$

and (4-16) can also be written in the form

$$P_D = P_{FA} + \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell [1 - S_{\ell+1}(\frac{a}{1+c})] . \quad (4-18)$$

The probability of detection when both threshold and signal parameter are random variables can, of course, be approached from two directions: we can start with random a and fixed y , then averaging over the assumed pdf of y , or proceed in the reverse order. It turns out that both methods yield useful formulas, not easily transformed into one other. We start with (4-16), averaging over the assumed pdf of a :

$$f_o(a) = \frac{1}{b} g_M(\frac{a}{b}) . \quad (4-19)$$

To carry this out we need the integral

$$\begin{aligned}
& \int_0^\infty s_{\ell+1}(\lambda x) g_{i_1}(x) dx \\
&= \sum_{m=0}^{\ell} \frac{\lambda^m}{m!} \int_0^\infty e^{-\lambda x} x^m g_{i_1}(x) dx \\
&= \sum_{m=0}^{\ell} \frac{\lambda^m}{m!} \frac{1}{(m-1)!} \int_0^\infty e^{-(1+\lambda)x} x^{m-1+m} dx \\
&= \frac{1}{(1+\lambda)^{\ell+1}} \sum_{m=0}^{\ell} \binom{m-1+m}{m} \left(\frac{\lambda}{1+\lambda}\right)^m. \tag{4-20}
\end{aligned}$$

This expression recurs in several of our formulas, and it should be noted that index ℓ occurs only as the upper limit to the sum. This fact will facilitate the recursive calculations of double sums, like (4-21) below.

When (4-16) is averaged over (4-19), we make the variable change, $a=bx$, to obtain

$$P_D = 1 - \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell \int_0^\infty s_{\ell+1}\left(\frac{bx}{1+c}\right) g_{i_1}(x) dx,$$

and now (4-20) provides the desired result

$$P_D = 1 - \frac{c^N}{(1+c)^{N+L-1}} \left(\frac{1+c}{1+b+c}\right)^M \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell \sum_{m=0}^{\ell} \binom{m-1+m}{m} \left(\frac{b}{1+b+c}\right)^m \tag{4-21}$$

Equation (4-21) is valid for all values of M , and it is particularly convenient for Swerling cases 1 and 3, where $M=1$ and 2, respectively. The false alarm probability is still given by (4-17).

For $M=1$, the m -sum is simply

$$\sum_{m=0}^L \left(\frac{b}{1+b+c} \right)^m = \frac{1+b+c}{1+c} \left\{ 1 - \left(\frac{b}{1+b+c} \right)^{L+1} \right\},$$

and (4-21) becomes

$$\begin{aligned} P_D &= 1 - \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+2} c^\ell \left\{ 1 - \left(\frac{b}{1+b+c} \right)^{L+1} \right\} \\ &= 1 - \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=N}^{N+L-1} \binom{N+L-1}{\ell} c^\ell \left\{ 1 - \left(\frac{b}{1+b+c} \right)^{L+1-N} \right\}. \end{aligned}$$

The ℓ -sum is now rewritten as the difference between a full sum (ℓ running from zero to $N+L-1$) and a partial sum, where ℓ runs from zero to $N-1$. The final result is ($M=1$):

$$P_D = \left(\frac{1+b}{b} \right)^{N-1} \left(\frac{1+b}{1+b+c} \right)^L + \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell \left\{ 1 - \left(\frac{1+b+c}{b} \right)^{N-1-\ell} \right\} \quad (4-22)$$

This formula is convenient because it has N terms, rather than L , and N is usually the smaller number. It can also be obtained from (4-12) by averaging over y . When $N=1$, (4-22) becomes simply

$$P_D = \left(\frac{1+b}{1+b+c} \right)^L . \quad (4-23)$$

In Swerling case 3 ($M=2$), the m -sum in (4-21) is

$$\sum_{m=0}^L (m+1)u^m = (1-u)^{-2} \left\{ 1 - (\ell+2)u^{\ell+1} + (\ell+1)u^{\ell+2} \right\} ,$$

where

$$u \equiv \frac{b}{1+b+c}$$

A calculation which parallels the derivation of (4-22) yields the result ($M=2$):

$$\begin{aligned} P_D = & \left(\frac{1+b}{b} \right)^{N-2} \left(\frac{1+b}{1+b+c} \right)^{L+1} \left\{ 1 - (N-2) \frac{1+c}{b} + (N+L-1) \frac{c}{1+b} \right\} \\ & + \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^{\ell} \left\{ 1 - [1 + (\ell+2-N) \frac{1+c}{b}] \left(\frac{1+b+c}{b} \right)^{N-2-\ell} \right\} . \end{aligned} \quad (4-24)$$

When $N=2$, this formula reduces to

$$P_D = \left(\frac{1+b}{1+b+c}\right)^{L+1} \left[1 + (L+1) \frac{c}{1+b}\right] . \quad (4-25)$$

When $M > N$, another formula is more convenient than (4-21). It is obtained by starting with (4-4) and randomizing y , thus reversing the order of randomization that led to (4-21). The pdf of y will be

$$f_o(y) = \frac{1}{c} g_L\left(\frac{y}{c}\right),$$

and we get

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell \int_0^\infty s_{N+\ell}\left(\frac{y}{1+b}\right) g_L\left(\frac{y}{c}\right) d\left(\frac{y}{c}\right) .$$

We change variables ($y=cx$) and use (4-20) to find

$$P_D = \left(\frac{1+b}{1+b+c}\right)^L \frac{1}{(1+b)^{M-N}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell \sum_{m=0}^{N+\ell-1} \binom{L-1+m}{m} \left(\frac{c}{1+b+c}\right)^m \quad (4-26)$$

The false alarm probability is still given by (4-17), and the fact that (4-26) reduces to (4-17) when $b=0$ rests upon the useful binomial identity

$$\sum_{m=0}^n \binom{L-1+m}{m} y^m = (1-y)^n \sum_{m=0}^n \binom{L+n}{m} \left(\frac{y}{1-y}\right)^m , \quad (4-27)$$

which is proved in Appendix 1.

When $M=N$ (Swerling 2), (4-26) becomes

$$P_D = \left(\frac{1+b}{1+b+c}\right)^L \sum_{m=0}^{N-1} \binom{L-1+m}{m} \left(\frac{c}{1+b+c}\right)^m . \quad (4-28)$$

For $N=1$, Swerling cases 1 and 2 coincide, and (4-28) becomes (4-23) in this case. Likewise, when $N=2$, cases 2 and 3 coincide, and (4-28) then reduces (4-25), as it should.

For Swerling case 4, $M=2N$, (4-26) becomes

$$P_D = \left(\frac{1+b}{1+b+c}\right)^L \frac{1}{(1+b)^N} \sum_{k=0}^N \binom{N}{k} b^k \sum_{m=0}^{N+k-1} \binom{L-1+m}{m} \left(\frac{c}{1+b+c}\right)^m . \quad (4-29)$$

V. GENERALIZATION OF THE FLUCTUATION MODELS

In Section 4, when a parameter, say y , was randomized, a pdf of the type

$$\frac{1}{c} g_L \left(\frac{y}{c}\right)$$

was used, introducing one new parameter. In this Section we generalize by using, instead, a "signal-plus-noise" type of pdf, such as

$$\frac{1}{c} f_L \left(\frac{y}{c}, d\right) .$$

This generalizes the chi-squared pdf to a non-central chi-squared, and adds

another parameter. The model is equivalent to the postulate $y = cx$, where x has the pdf $f_L(x, d)$ as defined by Eq. 3-2.

We begin by randomizing the signal parameter, a . We put $a = bx$, where x has pdf $f_M(x, d)$. In other words

$$f_o(a) = \frac{1}{b} f_M\left(\frac{a}{b}, d\right) \quad (5-1)$$

and the characteristic function is

$$\phi_o(\lambda) = E e^{i\lambda bx} = e^{-d} \frac{e^{\frac{d}{1-ib\lambda}}}{(1-ib\lambda)^M},$$

according to (3-4). To apply (3-18), we require

$$\phi_o[i(1 - \frac{1}{t})] = e^{-d} \left[\frac{t}{(1+b)t-b} \right]^M \exp \left[\frac{td}{(1+b)t-b} \right],$$

and then

$$p_D = e^{-y-d} \frac{1}{2\pi i} \int_{|t|=1-\epsilon} e^{yt} \exp \left[\frac{td}{(1+b)t-b} \right] \frac{t^{M-N} dt}{[(1+b)t-b]^M (1-t)}. \quad (5-2)$$

We confine ourselves to the case $M > N$ and, following the previous analysis, make the change of variable (4-3). The new feature is the exponential factor, which becomes

$$\exp\left[\frac{td}{(1+b)t-b}\right] = \exp\left(\frac{d}{s} \frac{s+b}{1+b}\right) = \exp\left(\frac{d}{1+b}\right) \exp\left(\frac{1}{s} \frac{bd}{1+b}\right) .$$

With the definitions

$$\bar{d} = \frac{bd}{1+b} , \quad \bar{y} = \frac{y}{1+b}$$

we have

$$P_D = e^{-\bar{y}-\bar{d}} \frac{1}{2\pi i} \int_{|s|=\epsilon} e^{\bar{y}s + \frac{\bar{d}}{s}} \left(\frac{s+b}{1+b}\right)^{M-N} \frac{ds}{s^M (1-s)} .$$

As before, we expand the factor $(s+b)^{M-N}$, and, this time, evaluate the integrals using (3-7). The result is

$$P_D = \frac{1}{(1+b)^{M-N}} \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} b^\ell P_{N+\ell}\left(\frac{y}{1+b}, \frac{bd}{1+b}\right) , \quad (5-3)$$

a direct generalization of (4-4), to which it reduces when $b=0$. When $b=0$, only the term $\ell=0$ remains, and the false alarm probability reduces to the expected value

$$P_{FA} = S_N(y), \quad (5-4)$$

The analog of Swerling's case 2 ($M=N$) is the simple expression

$$P_D = P_N\left(\frac{y}{1+b}, \frac{bd}{1+b}\right). \quad (5-5)$$

A result of this kind is to be expected (when $M=N$), since each complex sample, z_n , is now being modeled in the form

$$z_n = s_n + w_n,$$

where

$$s_n = \sqrt{b} (s'_n + w'_n).$$

The new "signal components," s'_n , satisfy

$$\frac{|s'_n|^2}{2\sigma^2} = d$$

while

$$E|w'_n|^2 = E|w_n|^2 = 2\sigma^2.$$

As a consequence,

$$z_n = \sqrt{1+b} (s_n'' + w_n'')$$

where

$$E|w_n''|^2 = 2\sigma^2$$

and

$$\frac{|s_n''|^2}{2\sigma^2} = \frac{b}{1+b} d.$$

Equation (5-5) then follows, since the new effective threshold is $y/(1+b)$.

In the dual problem, the signal parameter, a , is fixed, while the threshold is randomized by the statement: $y = cx$, when the pdf of x is $f_L(x, d)$. Then

$$f_o(y) = \frac{1}{c} f_L\left(\frac{y}{c}, d\right) \quad (5-6)$$

and the detection probability is obtained from (3-11) and (5-3) by making the changes of variable

$N \rightarrow 1 - N$
 $M \rightarrow L$
 $b \rightarrow c$
 $y \rightarrow a$.

Parameter d is unchanged, since it plays the same role in both problems. We obtain

$$P_D = 1 - \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} c^\ell P_{\ell+1-N} \left(\frac{a}{1+c}, \frac{cd}{1+c} \right), \quad (5-7)$$

which is a direct generalization of (4-16). Expression (5-7) reduces to (4-16) when $d=0$, since the terms $\ell=0$ through $\ell=N-1$ will vanish in that limit.

Although (5-3) contained only P -functions with positive subscripts, (5-7) involves negative subscripts as well. The latter are evaluated by means of (3-9), but first it is useful to apply (3-9) to all the terms of (5-7), to obtain the formula

$$P_D = \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} c^\ell P_{N-\ell} \left(\frac{cd}{1+c}, \frac{a}{1+c} \right) \quad (5-8)$$

Formula (5-8) is the basis of later generalizations, since it is more compact than (5-7) and also contains the parameters a and d , in a "natural order."

For computation, we split the terms with positive and non-positive subscripts, using (3-9) on the latter:

$$P_D = \frac{1}{(1+c)^{N+L-1}} \left\{ \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell P_{N-\ell} \left(\frac{cd}{1+c}, \frac{a}{1+c} \right) \right. \\ \left. + \sum_{\ell=N}^{N+L-1} \binom{N+L-1}{\ell} c^\ell \left[1 - P_{\ell+1-N} \left(\frac{a}{1+c}, \frac{cd}{1+c} \right) \right] \right\} \quad (5-9)$$

Finally, the first sum in (5-9) is reversed, and the index redefined in the second sum to obtain the result

$$\begin{aligned}
 P_D = & \frac{c^N}{(1+c)^{N+L-1}} \left\{ \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c}\right)^{\ell+1} P_{\ell+1} \left(\frac{cd}{1+c}, \frac{a}{1+c}\right) \right. \\
 & \left. + \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell [1 - P_{\ell+1} \left(\frac{a}{1+c}, \frac{cd}{1+c}\right)] \right\} \quad (5-10)
 \end{aligned}$$

To obtain the false alarm probability, we put $a=0$ in (5-10) and note that

$$P_{\ell+1} \left(0, \frac{cd}{1+c}\right) = 1, \quad \text{and}$$

$$P_{\ell+1} \left(\frac{cd}{1+c}, 0\right) = S_{\ell+1} \left(\frac{cd}{1+c}\right);$$

therefore

$$P_{FA} = \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c}\right)^{\ell+1} S_{\ell+1} \left(\frac{cd}{1+c}\right). \quad (5-11)$$

which reduces to (4-17) when $d=0$.

As mentioned in Section 1, the present model describes the performance of a linear CFAR system with a non-fluctuating signal and with other signal components in the terms used to form the detection threshold. These signal components are usually unwanted, and they can strongly modify the performance of a detection system. One obvious effect of these signal contributions is to reduce the false alarm probability (by increasing the threshold), which can be seen from (5-11). All the terms in that sum are positive, and $S_{\ell+1} [cd/(1+c)]$ is a decreasing function of d , since $S_n(y)$ is the false alarm probability of a system with n samples and threshold y , which decreases with increasing y . The linear "CFAR" loses its CFAR property with this change in the statistical properties of the noise. It is evident from (5-10) that P_D is reduced by these signal components also.

Since P_D is the probability that $u > cx$, we have

$$1-P_D = \text{Prob} \left[\frac{u}{x} < c \right].$$

Thus $1-P_D$ is the cumulative probability distribution of the "non-central F" random variable, $\rho \equiv u/x$. Both numerator and denominator of this ratio are non-central chi-squared variables, and (5-10) provides a basis for computation which is more convenient than the standard series, as given by Springer.(10) When $d=0$, the variable x becomes simply chi-squared, and the resulting cumulative probability distribution is a finite sum of elementary functions, given by (4-16). It should be noted that all the chi-squared variables discussed here are of even order, and this accounts for the relative simplicity of the formulas.

We now extend our results by randomizing a and then d , finally treating the case where both a and d are random variables. We begin with a , and postulate for this parameter the pdf

$$f_0(a) = \frac{1}{b} g_M\left(\frac{a}{b}\right),$$

as before. Both sides of (5-8) are multiplied by $f_0(a)$, and integration over a is carried out after the change of variable: $a=(1+c)x$. We get

$$P_D = \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} c^\ell \int_0^\infty p_{N-\ell} \left(\frac{cd}{1+c}, x\right) g_M\left(\frac{1+c}{b} x\right) \frac{1+c}{b} dx. \quad (5-12)$$

The integral is the probability of detection for a fixed threshold and fluctuating signal parameter, and hence it is expressible by means of the results of Section 4 with the following identification of variables:

$$N \rightarrow N-\ell$$

$$y \rightarrow \frac{cd}{1+c}$$

$$b \rightarrow \frac{b}{1+c}$$

(5-13)

However, the sum in (5-12) may have to be separated into two parts, where $N-\ell \geq M$ in one part, and $N-\ell < M$ in the other, unless $M > N$, in which case all terms are of the same type. We carry out this procedure only for the case $M=1$ (Swerling case 1), before making the assumption that $M > N$.

When $M=1$, we write (5-12) in the form

$$P_D = \frac{1}{(1+c)^{N+L-1}} \left\{ \sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell X(\ell) \right.$$

$$+ \sum_{\ell=N}^{N+L-1} \binom{N+L-1}{\ell} c^\ell Y(\ell) \Big\} , \quad (5-14)$$

and use (4-12) to evaluate $X(\ell)$, and (4-4) for $Y(\ell)$. First, we recall that (4-12) itself can be written

$$P_D = S_N(y) + \left(\frac{1+b}{b}\right)^{N-1} e^{-\frac{y}{1+b}} \left[1 - S_N\left(\frac{by}{1+b}\right)\right], \quad (4-12)$$

which is valid for all $N \geq 1$. When the variable substitutions (5-13) are made, we find from this version of (4-12) that

$$X(\ell) = S_{N-\ell}\left(\frac{cd}{1+c}\right) + \left(\frac{1+b+c}{b}\right)^{N-\ell-1} e^{-\frac{cd}{1+b+c}} \left[1 - S_{N-\ell}\left(\frac{b}{1+c} \frac{cd}{1+b+c}\right)\right] .$$

In the corresponding sum over ℓ , we write

$$\sum_{\ell=0}^{N-1} \binom{N+L-1}{\ell} c^\ell X(\ell) = \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} c^{N-1-\ell} X(N-1-\ell)$$

to obtain an increasing sequence of S-functions. Then,

$$\begin{aligned}
 p_D = & \frac{c^{N-1}}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c}\right)^\ell \left\{ S_{\ell+1} \left(\frac{cd}{1+c}\right) + \right. \\
 & \left. + \left(\frac{1+b+c}{b}\right)^\ell e^{-\frac{cd}{1+b+c}} \left[1 - S_{\ell+1} \left(\frac{b}{1+c} - \frac{cd}{1+b+c}\right) \right] \right\} \\
 & + \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=N}^{N+L-1} \binom{N+L-1}{\ell} c^\ell Y(\ell) . \tag{5-15}
 \end{aligned}$$

To obtain $Y(\ell)$, we use (4-4) with the substitutions (5-13) and $M=1$:

$$Y(\ell) = \left(\frac{1+c}{1+b+c}\right)^{1-N+\ell} \sum_{m=0}^{1-N+\ell} \binom{\ell}{m} \left(\frac{b}{1+c}\right)^m S_{N+m-\ell} \left(\frac{cd}{1+b+c}\right) .$$

In (5-14), $Y(\ell)$ enters only in terms where $\ell > N$, and also, the S -function is zero unless $m > \ell - N$. This occurs in only one term of the m -sum (the upper limit) hence

$$Y(\ell) = \left(\frac{b}{1+b+c}\right)^{1-N+\ell} S_1 \left(\frac{cd}{1+b+c}\right) .$$

Of course, $S_1(x) = e^{-x}$, and we obtain

$$\begin{aligned}
P_D &= \frac{c^{N-1}}{(1+c)^{N+L-1}} \left\{ \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1}{c}\right)^\ell S_{\ell+1} \left(\frac{cd}{1+c}\right) \right. \\
&+ e^{-\frac{cd}{1+b+c}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1+b+c}{bc}\right)^\ell \left[1 - S_{\ell+1} \left(\frac{b}{1+c} \frac{cd}{1+b+c}\right) \right] \\
&+ \left. e^{-\frac{cd}{1+b+c}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} \left(\frac{bc}{1+b+c}\right)^{\ell+1} \right\}. \tag{5-16}
\end{aligned}$$

The terms not involving S-functions in (5-16) can be combined, as follows

$$\begin{aligned}
&\sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left(\frac{1+b+c}{bc}\right)^\ell + \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} \left(\frac{bc}{1+b+c}\right)^{\ell+1} \\
&= \sum_{\ell=L}^{N+L-1} \binom{N+L-1}{\ell} \left(\frac{1+b+c}{bc}\right)^{\ell-L} + \sum_{\ell=0}^{L-1} \binom{N+L-1}{\ell} \left(\frac{bc}{1+b+c}\right)^{L-\ell} \\
&= \left(\frac{bc}{1+b+c}\right)^L \left(1 + \frac{1+b+c}{bc}\right)^{N+L-1} \\
&= \frac{[(1+b)(1+c)]^{N+L-1}}{(bc)^{N-1}(1+b+c)^L}.
\end{aligned}$$

Therefore finally,

$$\begin{aligned}
 P_D = & \left(\frac{1+b}{b} \right)^{N-1} \left(\frac{1+b}{1+b+c} \right)^L e^{-\frac{cd}{1+b+c}} \\
 & + \frac{c^{N-1}}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N-1} \binom{N+L-1}{L+\ell} \left\{ \left(\frac{1}{c} \right)^\ell S_{\ell+1} \left(\frac{cd}{1+c} \right) \right. \\
 & \left. - e^{-\frac{cd}{1+b+c}} \left(\frac{1+b+c}{bc} \right)^\ell S_{\ell+1} \left(\frac{b}{1+c} \frac{cd}{1+b+c} \right) \right\} \quad (5-17)
 \end{aligned}$$

The corresponding false alarm probability is still given by (5-11), and it can be shown directly that (5-17) converges to (5-11) as $b \rightarrow 0$, and to (4-22) as $d \rightarrow 0$.

We forego a similar analysis of the Swerling case 3 problem and return to (5-12) with the assumption that $M > N$. Then (4-4) can be used for all the terms, and we obtain

$$\begin{aligned}
 P_D = & \frac{1}{(1+c)^{N+L-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} c^\ell x \\
 & \times \left(\frac{1+c}{1+b+c} \right)^{M-N+\ell} \sum_{m=0}^{M-N+\ell} \binom{M-N+\ell}{m} \left(\frac{b}{1+c} \right)^m S_{N+m-\ell} \left(\frac{cd}{1+b+c} \right) . \quad (5-18)
 \end{aligned}$$

It is useful to rearrange the sums in (5-18) in order to get an increasing sequence of S-functions (to permit their recursive computation), hence we write $N+m-\ell = n+1$, and eliminate m . The lower limit on the new index, n , is set by the S-functions themselves at $n=0$, and the upper limit is seen to be $M-1$. In reordering these sums, we note that the original sums over m and ℓ are automatically limited by their binomial coefficients, since $\binom{n}{k}$ is zero if either k or $n-k$ is negative. Thus, P_D can be written

$$P_D = F_0 \sum_{n=0}^{M-1} \left\{ \sum_{\ell} \binom{N+L-1}{\ell} \binom{M-N+\ell}{n+1+\ell-N} x^{\ell} \right\} \left(\frac{b}{1+c} \right)^n S_{n+1} \left(\frac{cd}{1+b+c} \right) , \quad (5-19)$$

where

$$F_0 \equiv \frac{(1+c)^{M-N-L}}{b^{N-1} (1+b+c)^{M-N}} ,$$

and

$$x \equiv \frac{bc}{1+b+c} .$$

The lower limit of the ℓ -sum in (5-19) depends upon n .

Next, we use the identity

$$\sum_{\ell} \binom{A}{\ell} \binom{B+\ell}{C} x^{\ell} = x^{C-B} (1+x)^{A-C} \sum_{\ell} \binom{B}{\ell} \binom{A+\ell}{A+B-C} x^{\ell} , \quad (5-20)$$

which is proved in Appendix 1. The sums on both sides of (5-20) are understood to be limited automatically by the corresponding binomial coefficients. To apply (5-20), we note that

$$\binom{N-N+l}{n+l+2-N} = \binom{N-N+l}{N-n-1} \quad ,$$

and then we have

$$\sum_l \binom{N+L-1}{l} \binom{M-N+l}{N-n-1} x^l \\ = x^{N-n-1} (1+x)^{N+L-M+n} \sum_l \binom{M-N}{l} \binom{N+L-1+l}{L-n} x^l \quad .$$

We find that

$$F_0 x^{N-1} (1+x)^{N+L-M} = \frac{c^{N-1} (1+b)^{N+L-M}}{(1+b+c)^{N+L-1}} \quad ,$$

and that

$$\left(\frac{1+x}{x}\right)^n \left(\frac{b}{1+c}\right)^n = \left(\frac{1+b}{c}\right)^n \quad .$$

When these evaluations are substituted into (5-19), the order of summation can be reversed, with the result

$$P_D = \frac{c^{N-1}(1+b)^{N+L-M}}{(1+b+c)^{N+L-1}} \sum_{\lambda=0}^{M-N} \binom{M-N}{\lambda} \left(\frac{bc}{1+b+c}\right)^\lambda x$$

$$x \sum_{n=0}^{N-1+\lambda} \binom{N+L-1+\lambda}{L+n} \left(\frac{1+b}{c}\right)^n S_{n+1} \left(\frac{cd}{1+b+c}\right) . \quad (5-21)$$

In Swerling case 2 ($M=N$), the λ -sum in (5-21) reduces a single term; and

$$P_D = \frac{c^{N-1}(1+b)^L}{(1+b+c)^{N+L-1}} \sum_{n=0}^{N-1} \binom{N+L-1}{L+n} \left(\frac{1+b}{c}\right)^n S_{n+1} \left(\frac{cd}{1+b+c}\right) . \quad (5-22)$$

The false alarm probability for any M is still given by (5-11), to which (5-21) reduces when $b=0$.

For Swerling case 4, (5-21) must be used with $M=2N$. It should be noted that (5-21) can be obtained somewhat more directly by averaging (4-4) over the pdf,

$$f_o(y) = \frac{1}{c} f_L\left(\frac{y}{c}, d\right) , \quad (5-23)$$

for the threshold y . The required integrals,

$$\int_0^\infty S_{\lambda+1}(\lambda x) f_L(x, d) dx,$$

are evaluated by writing

$$f_L(x, d) = -\frac{d}{dx} P_L(x, d),$$

then integrating by parts, and using (4-16).

So far, we have been discussing extensions of (5-8) in which the signal parameter, a , has been randomized. In the dual problem, the parameter d is randomized instead. This represents linear CFAR with a fixed signal parameter, a , but with fluctuating signal components in the randomized threshold. For d we assume the pdf

$$f_o(d) = \frac{1}{h} g_k(\frac{d}{h}), \quad (5-24)$$

and average (5-8) with respect to it. The result is

$$P_D = \frac{1}{(1+c)^{N+L-1}} \sum_{\lambda=0}^{N+L-1} \binom{N+L-1}{\lambda} c^\lambda \int_0^\infty P_{N-\lambda}(y, \frac{a}{1+c}) g_K(\frac{1+c}{ch} y) \frac{1+c}{ch} dy \quad (5-25)$$

after the change of variable $d+(1+c)y/c$. We can use (4-16) to evaluate the integral, which corresponds to ordinary linear CFAR and fixed signal parameter, with the variable changes:

$N \rightarrow N-\lambda$

$L \rightarrow K$

$$a \rightarrow \frac{a}{1+c}$$

$$c \rightarrow \frac{ch}{1+c}$$

Therefore

$$0 \int^{\infty} P_{N-\ell}(y, \frac{a}{1+c}) g_K(\frac{1+c}{ch} y) \frac{1+c}{ch} dy$$

$$= 1 - \left(\frac{1+c}{1+c+ch}\right)^{N+K-\ell-1} \sum_{m=N-\ell}^{N+K-\ell-1} \binom{N+K-\ell-1}{m} \left(\frac{ch}{1+c}\right)^m s_{m+\ell+1-N} \left(\frac{a}{1+c+ch}\right).$$

We assume that $K > L$ to assure the validity of this formula for all ℓ (in particular, $\ell = N+L-1$). When (4-16) was derived, the parameter there called N was non-negative, and we are now applying to a more general case. A review of the derivation of (4-16) will show that it is valid as long as $N+L-1 > 0$, which explains the requirement $K > L$. This excludes Swerling case 1 type models for the fluctuating signal component in the threshold.

Substituting in (5-25) we get

$$P_D = 1 - \frac{(1+c)^{K-L}}{(1+c+ch)^{N+K-1}} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} \left[c \frac{1+c+ch}{1+c}\right]^{\ell} x$$

$$x \sum_{m=0}^{N+K-\ell-1} \binom{N+K-\ell-1}{m} \left(\frac{ch}{1+c}\right)^m s_{m+\ell+1-N} \left(\frac{a}{1+c+ch}\right).$$

The sum on m has been allowed to start at $m=0$, since the S -function will vanish in the extra terms. With substitution $\ell+N+L-1-\ell$, this becomes

$$P_D = 1 - \left(\frac{c}{1+c} \right)^{N+L-1} \left(\frac{1+c}{1+c+ch} \right)^{K-L} \sum_{\ell=0}^{N+L-1} \binom{N+L-1}{\ell} \left[\frac{1}{c} \frac{1+c}{1+c+ch} \right]^{\ell} \\ \times \sum_{m=0}^{K-L+\ell} \binom{K-L+\ell}{m} \left(\frac{ch}{1+c} \right)^m S_{L+m-\ell} \left(\frac{a}{1+c+ch} \right) . \quad (5-26)$$

Equation (5-26) can be obtained from (5-18) by reversing the roles of the original random variables, complimenting the probability, and making the parameter substitutions

$$\begin{array}{ll} N+L & b+h \\ L+N & d+a \\ M+K & c+1/c \end{array} . \quad (5-27)$$

A form analogous to (5-21) can be obtained from (5-26), either by a parallel calculation or by making the variable substitutions (5-29) in (5-21). Either way, the result is

$$P_D = 1 - \frac{c^N (1+h)^{N+L-K}}{(1+c+ch)^{N+L-1}} \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch} \right)^{\ell} x \\ \times \sum_{n=0}^{L-1+\ell} \binom{N+L-1+\ell}{N+n} (c+ch)^n S_{n+1} \left(\frac{a}{1+c+ch} \right) . \quad (5-28)$$

It can be seen that (5-28) reduces to (4-16) when $h=0$, since then only the term $\ell=0$ survives.

The false alarm probability is obtained from (5-28) by putting $a=0$, which has only the effect of replacing the S-function by unity. But then,

$$\sum_{n=0}^{L-1+\ell} \binom{N+L-1+\ell}{N+n} (c+ch)^n = (c+ch)^{-N} \sum_{n=N}^{N+L-1+\ell} \binom{N+L-1+\ell}{n} (c+ch)^n$$

$$= \frac{(1+c+ch)^{N+L-1+\ell}}{(c+ch)^N} - \frac{1}{(c+ch)^N} \sum_{n=0}^{N-1} \binom{N+L-1+\ell}{n} (c+ch)^n, \quad (5-29)$$

and

$$\sum_{\ell=0}^{K-L} \binom{K-L}{\ell} h^{\ell} = (1+h)^{K-L}.$$

The result is that the first term of (5-29) eventually cancels the unity in (5-28) and the false alarm probability simplifies to

$$p_{FA} = \frac{(1+h)^{L-K}}{(1+c+ch)^{N+L-1}} \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch}\right)^{\ell} \times$$

$$\times \sum_{n=0}^{N-1} \binom{N+L-1+\ell}{n} (c+ch)^n. \quad (5-30)$$

This expression reduces, of course, to (4-17) when $h=0$.

The final situation, in which both of the signal parameters, a and d , are randomized is analyzed first by averaging (5-28) with respect to the usual pdf for a :

$$f_o(a) = \frac{1}{b} g_M\left(\frac{a}{b}\right).$$

In this way, a formula valid for all M is obtained, although the constraint $K > L$ is still in force. Only the S -function in (5-28) depends on a , and the use of (4-20) gives us

$$\begin{aligned} & \int_0^\infty S_{n+1}\left(\frac{a}{1+c+ch}\right) g_M\left(\frac{a}{b}\right) d\left(\frac{a}{b}\right) \\ &= \left(\frac{1+c+ch}{1+b+c+ch}\right)^M \sum_{m=0}^n \binom{M-1+m}{m} \left(\frac{b}{1+b+c+ch}\right)^m. \end{aligned} \quad (5-31)$$

When (5-31) is used to average (5-28), the result is

$$\begin{aligned} P_D &= 1 - \frac{c^N (1+h)^{N+L-K} (1+c+ch)^{M-N-L+1}}{(1+b+c+ch)^M} x \\ &\times \sum_{\ell=0}^{K-L} \binom{K-L}{\ell} \left(\frac{h}{1+c+ch}\right)^\ell \sum_{n=0}^{L-1+\ell} \binom{N+L-1+\ell}{N+n} (c+ch)^n x \end{aligned}$$

$$x \sum_{m=0}^n \binom{M-1+m}{m} \left(\frac{b}{1+b+c+ch}\right)^m , \quad (5-32)$$

and the probability of false alarm is still given by (5-30).

To reverse the constraints, we can obtain a formula valid for all K and $M > N$ by averaging (5-21) with respect to the pdf of d , namely (5-24). The resulting detection formula is

$$p_D = \frac{c^{N-1} (1+b)^{N+L-M} (1+b+c)^{K-N-L+1}}{(1+b+c+ch)^K} x$$

$$x \sum_{\ell=0}^{M-N} \binom{M-N}{\ell} \left(\frac{bc}{1+b+c}\right)^\ell \sum_{n=0}^{N-1+\ell} \binom{N+L-1+\ell}{L+n} \left(\frac{1+b}{c}\right)^n x$$

$$x \sum_{m=0}^n \binom{K-1+m}{m} \left(\frac{ch}{1+b+c+ch}\right)^m . \quad (5-33)$$

Expression (5-33) reduces to (4-26) when $h=0$, as does (5-21) in the corresponding limit, $d=0$. It is possible to show that (5-32) and (5-33) are equivalent when both $K > L$ and $M > N$.

The false alarm probability corresponds to $b=0$:

$$\begin{aligned}
 P_{FA} = & \frac{c^{N-1}(1+c)^{K-N-L+1}}{(1+c+ch)^K} \sum_{n=0}^{N-1} \binom{N+L-1}{L+n} \left(\frac{1}{c}\right)^n \times \\
 & \times \sum_{m=0}^n \binom{K-1+m}{m} \left(\frac{ch}{1+c+ch}\right)^m . \quad (5-34)
 \end{aligned}$$

We did not work out the detection and false alarm probabilities for the analogous fixed-signal case, with $K < L$, but (5-34) is identical to the result of averaging (5-11) over the pdf (5-24) for d .

Formulas (5-32) and (5-33) generalize some results of Rickard and Dillard,⁽¹¹⁾ and also Wishner's analysis⁽¹⁸⁾ of the normalized periodogram detector.

VI. RELATION TO OTHER FORMS

A striking feature of the finite sum detection formulas is the diversity of equivalent forms they can assume. Solutions to the same problem, such as (4-21) and (4-26), deduced from different starting points can be reconciled only with difficulty. In this section, we relate a few of our results to quite different formulas which have appeared in the literature.

Consider Eq. (4-16) which describes linear CFAR with a fixed signal parameter, a . It is

$$1 - P_D = \frac{c^N}{(1+c)^{N+L-1}} \sum_{\ell=0}^{L-1} \binom{N+L-1}{N+\ell} c^\ell s_{\ell+1} \left(\frac{a}{1+c}\right) , \quad (6-1)$$

which can be written

$$\left(\frac{1+c}{c}\right)^N (1 - p_D) \equiv F_N(L-1) = \sum_{\ell=0}^{L-1} T_N(L-1, \ell) \quad (6-2)$$

where

$$T_N(L, \ell) \equiv \frac{1}{(1+c)^L} \binom{N+L}{N+\ell} c^\ell S_{\ell+1}\left(\frac{a}{1+c}\right) \quad (6-3)$$

Now $F_N(L)$ can be rewritten as a sum of terms, each of which depends only on ℓ , and not also on L , as do the terms $T_N(L, \ell)$. To accomplish this we write

$$F_N(L) = \sum_{\ell=0}^L T_N(L, \ell) = T_N(L, L) + \sum_{\ell=0}^{L-1} T_N(L, \ell)$$

$$= T_N(L, L) + \sum_{\ell=0}^{L-1} [T_N(L, \ell) - T_N(L-1, \ell)] + F_N(L-1),$$

or

$$F_N(L) = A_N(L) + F_N(L-1), \quad (6-4)$$

where

$$A_N(L) \equiv T_N(L, L) + \sum_{\ell=0}^{L-1} [T_N(L, \ell) - T_N(L-1, \ell)] \quad (6-5)$$

for $L \geq 1$. Iterating (6-4) we obtain

$$F_N(L) = A_N(L) + A_N(L-1) + \dots + A_N(1) + F_N(0) .$$

Defining

$$A_N(0) \equiv F_N(0) = T_N(0,0) , \quad (6-6)$$

we can write

$$F_N(L) = \sum_{\ell=0}^L A_N(\ell) , \quad (6-7)$$

which has the desired characteristic.

As applied to (6-2), we have

$$T_N(L, \ell) = \frac{1}{(1+c)^L} \binom{N+L}{N+\ell} c^\ell S_{\ell+1}(u) , \quad (6-8)$$

where

$$u \equiv \frac{a}{1+c} .$$

One finds that

$$T_N(0,0) = S_1(u) = e^{-u} ,$$

$$T_N(L,L) = \left(\frac{c}{1+c}\right)^L S_{L+1}(u) ,$$

and

$$T_N(L,\ell) = T_N(L-1,\ell)$$

$$= \frac{1}{(1+c)^L} \left\{ \binom{N+L}{N+\ell} - (1+c) \binom{N+L-1}{N+\ell} \right\} c^\ell S_{\ell+1}(u)$$

$$= \frac{1}{(1+c)^L} \left\{ \binom{N+L-1}{N+\ell-1} - c \binom{N+L-1}{N+\ell} \right\} c^\ell S_{\ell+1}(u) .$$

The standard reduction formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

has been used here. When these results are combined, we get

$$(1+c)^L A_N(L) = \sum_{\ell=1}^L \binom{N+L-1}{N+\ell-1} c^\ell [S_{\ell+1}(u) - S_\ell(u)] + \binom{N+L-1}{N-1} S_1(u)$$

$$= e^{-u} \sum_{\ell=0}^L \binom{N+L-1}{N+\ell-1} \frac{(cu)^\ell}{\ell!} \quad (6-9)$$

which holds also for $L=0$. Therefore

$$F_N(L) = e^{-u} \sum_{\ell=0}^L \frac{1}{(1+c)^\ell} \sum_{m=0}^\ell \binom{N+\ell-1}{N+m-1} \frac{(cu)^m}{m!} \quad ,$$

and L now appears only as the upper limit of the outer sum.

The finite m -sum is a special case of the confluent hypergeometric function; in fact it is a Laguerre polynomial.⁽¹²⁾ The basic definition is

$$L_m^{(k)}(x) = \sum_{n=0}^m \binom{k+n}{k} \frac{(-x)^n}{n!} \quad , \quad (6-10)$$

hence

$$F_N(L) = e^{-u} \sum_{\ell=0}^L \frac{1}{(1+c)^\ell} L_\ell^{(N-1)}(-cu) \quad ,$$

and finally

$$P_D = 1 - \left(\frac{c}{1+c}\right)^N e^{-\frac{a}{1+c}} \sum_{i=0}^{L-1} \frac{1}{(1+c)^i} L_i^{(N-i)}\left(-\frac{ca}{1+c}\right) \quad . \quad (6-11)$$

The Laguerre polynomials can be computed from the second-order recursion relation⁽¹²⁾

$$L_{\ell}^{(k)}(-x) = \left(2 + \frac{k-1+x}{\ell}\right) L_{\ell-1}^{(k)}(-x) - \left(1 + \frac{k-1}{\ell}\right) L_{\ell-2}^{(k)}(-x) \quad (6-12)$$

together with

$$L_0^{(k)}(-x) = 1 \quad (6-13)$$

$$L_1^{(k)}(-x) = k+1+x .$$

The particular case, $N=1$, of formula (6-11) was given by Finn,⁽¹³⁾ whose $L_{\ell}(x)$ equals $\ell! L_{\ell}^{(0)}(-x)$, in our notation.

It is interesting that the probability of false alarm obtained from (6-11) appears in the form

$$P_{FA} = 1 - \left(\frac{c}{1+c}\right)^N \sum_{\ell=0}^{L-1} \binom{N+\ell-1}{\ell} \frac{1}{(1+c)^\ell} , \quad (6-14)$$

which can be transformed into (4-17) by the use of (4-27).

Formula (4-4) can be treated the same way, when $M>N$. According to (4-4),

$$P_D = F_N(M-N),$$

where

$$F_N(K) = \frac{1}{(1+b)^K} \sum_{\ell=0}^K \binom{K}{\ell} b^\ell S_{N+\ell}(u) , \quad (6-15)$$

and

$$u \equiv \frac{y}{1+b} .$$

In this case

$$T_N(K, \ell) = \frac{1}{(1+b)^K} \binom{K}{\ell} b^\ell S_{N+\ell}(u) ,$$

and a calculation very similar to the one just given yields

$$A_N(K) = \frac{b}{(1+b)^K} e^{-u} \sum_{\ell=0}^{K-1} \binom{K-1}{\ell} b^\ell \frac{u^{N+\ell}}{(N+\ell)!} . \quad K \geq 0 ,$$

$$A_N(0) = S_N(u) .$$

In terms of Laguerre polynomials,

$$\sum_{\ell=0}^{K-1} \binom{K-1}{\ell} \frac{(bu)^\ell}{(N+\ell)!} = \frac{(K-1)!}{(N+K-1)!} \sum_{\ell=0}^{K-1} \binom{N+K-1}{N+\ell} \frac{(bu)^\ell}{\ell!} = \frac{(K-1)!}{(N+K-1)!} L_{K-1}^{(N)}(-bu) ,$$

hence

$$A_N(K) = \frac{bu^N}{(1+b)^K} e^{-u} \frac{(K-1)!}{(N+K-1)!} L_{K-1}^{(N)}(-bu) .$$

Finally,

$$F_N(K) = S_N(u) + \sum_{k=1}^K A_N(k) \\ = S_N(u) + bu^N e^{-u} \sum_{k=1}^K \frac{(k-1)!}{(N+k-1)!} \frac{1}{(1+b)^k} L_{k-1}^{(N)}(-bu) ,$$

and therefore

$$P_D = S_N\left(\frac{y}{1+b}\right) + \frac{b}{1+b} \left(\frac{y}{1+b}\right)^N e^{-\frac{y}{1+b}} \sum_{k=0}^{M-N-1} \frac{k!}{(N+k)!} \frac{1}{(1+b)^k} L_k^{(N)}\left(-\frac{by}{1+b}\right). \quad (6-16)$$

This time, the false alarm probability ($b=0$) is simply

$$P_{FA} = S_N(y) ,$$

the same as (4-5).

We demonstrate another kind of equivalence now, reducing an infinite series solution for the fixed-signal, linear CFAR problem to our finite sum

(4-16). The former solution is obtained by averaging Fehlner's' series:

$$P_N(y, a) = \int_{-a}^{\infty} \sum_{\ell=0}^{\infty} \frac{a^{\ell}}{\ell!} S_{N+\ell}(y) \, dy \quad (6-17)$$

over the pdf

$$f_0(y) = \frac{1}{c} g_L\left(\frac{y}{c}\right)$$

for the threshold. Series (6-17) can be obtained from (3-7) by expanding the factor $\exp(a/t)$ in a power series and integrating term by term. These integrals are performed by using (3-14).

According to (4-20),

$$\begin{aligned} \int_0^{\infty} S_{N+\ell}(y) g_L\left(\frac{y}{c}\right) d\left(\frac{y}{c}\right) &= \int_0^{\infty} S_{N+\ell}(cx) g_L(x) dx \\ &= \frac{1}{(1+c)^L} \sum_{m=0}^{N+\ell-1} \binom{L-1+m}{m} \left(\frac{c}{1+c}\right)^m \\ &= \frac{1}{(1+c)^{N+L+\ell-1}} \sum_{m=0}^{N+\ell-1} \binom{N+L+\ell-1}{m} c^m. \end{aligned} \quad (6-18)$$

The last step in the derivation of (6-18) is an application of (4-27).
 Another transformation yields

$$\begin{aligned}
 & \int_0^\infty s_{N+\ell}(y) g_L\left(\frac{y}{c}\right) d\left(\frac{y}{c}\right) \\
 &= 1 - \frac{1}{(1+c)^{N+L+\ell-1}} \sum_{m=N+\ell}^{N+L+\ell-1} \binom{N+L+\ell-1}{m} c^m \\
 &= 1 - \frac{c^{N+\ell}}{(1+c)^{N+L+\ell-1}} \sum_{m=0}^{L-1} \binom{N+L+\ell-1}{N+\ell+m} c^m . \tag{6-19}
 \end{aligned}$$

When (6-19) is used to average (6-17), the result is

$$\begin{aligned}
 1 - P_D &= e^{-a} \sum_{\ell=0}^{\infty} \frac{a^\ell}{\ell!} \frac{c^{N+\ell}}{(1+c)^{N+L+\ell-1}} \sum_{m=0}^{L-1} \binom{N+L+\ell-1}{N+\ell+m} c^m \\
 &= e^{-a} \frac{c^N}{(1+c)^{N+L-1}} \sum_{m=0}^{L-1} c^m \sum_{\ell=0}^{\infty} \binom{N+L+\ell-1}{N+\ell+m} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^\ell . \tag{6-20}
 \end{aligned}$$

But

$$\begin{aligned}
 & \sum_{\ell=0}^{\infty} \binom{N+L+\ell-1}{N+\ell+m} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^{\ell} \\
 &= \binom{N+L-1}{N+m} \sum_{\ell=0}^{\infty} \frac{(N+L-1+\ell)!(N+m)!}{(N+L-1)!(N+m+\ell)!} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^{\ell} \\
 &= \binom{N+L-1}{N+m} {}_1F_1(N+L; N+m+1; \frac{ca}{1+c}),
 \end{aligned}$$

in terms of the confluent hypergeometric function.⁽¹⁵⁾ Applying Kummer's first transformation,⁽¹⁶⁾

$$\begin{aligned}
 {}_1F_1(N+L; N+m+1; \frac{ca}{1+c}) &= e^{\frac{ca}{1+c}} {}_1F_1(m+1-L; N+m+1; -\frac{ca}{1+c}) \\
 &= e^{\frac{ca}{1+c}} \sum_{\ell=0}^{L-1-m} \frac{(L-1-m)!(N+m)!!}{(L-1-m-\ell)!(N+m+\ell)!} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^{\ell}.
 \end{aligned}$$

The new hypergeometric function is simply a polynomial, since its first argument, $m+1-L$, is a non-positive integer. Altogether,

$$\sum_{\ell=0}^{\infty} \binom{N+L+\ell-1}{N+\ell+m} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^\ell = e^{\frac{ca}{1+c}} \sum_{\ell=0}^{L-1-m} \binom{N+L-1}{N+m+\ell} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^\ell ,$$

and hence

$$1 - P_D = e^{-\frac{a}{1+c}} \frac{c^N}{(1+c)^{N+L-1}} \sum_{m=0}^{L-1} c^m \sum_{\ell=0}^{L-1-m} \binom{N+L-1}{N+m+\ell} \frac{1}{\ell!} \left(\frac{ca}{1+c}\right)^\ell \quad (6-21)$$

Next, we replace m by the new index $n = m+\ell$, and find

$$\begin{aligned} 1 - P_D &= e^{-\frac{a}{1+c}} \frac{c^N}{(1+c)^{N+L-1}} \sum_{n=0}^{L-1} \binom{N+L-1}{N+n} c^n \sum_{\ell=0}^n \frac{1}{\ell!} \left(\frac{a}{1+c}\right)^\ell \\ &= \frac{c^N}{(1+c)^{N+L-1}} \sum_{n=0}^{L-1} \binom{N+L-1}{N+n} c^n S_{n+1} \left(\frac{a}{1+c}\right) , \end{aligned}$$

which is (6-16). Similarly by averaging (6-17) over a , using the usual pdf with $M > N$, (4-4) can be derived.

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APPENDIX 1

The binomial identity

$$\sum_{m=0}^n \binom{M-1+m}{m} x^m = (1-x)^n \sum_{m=0}^n \binom{M+n}{m} \left(\frac{x}{1-x}\right)^m \quad (A-1)$$

is proved by taking the factor $(1-x)^n$ inside the sum and expanding again, as follows:

$$(1-x)^n \sum_{m=0}^n \binom{M+n}{m} \left(\frac{x}{1-x}\right)^m = \sum_{m=0}^n \binom{M+n}{m} x^m (1-x)^{n-m}$$

$$= \sum_{m=0}^n \sum_{k=0}^{n-m} \binom{M+n}{m} \binom{n-m}{k} (-1)^k x^{m+k}$$

$$= \sum_{\ell=0}^n \left\{ \sum_{m=0}^{\ell} \binom{M+n}{m} \binom{n-m}{\ell-m} (-1)^{\ell-m} \right\} x^{\ell} .$$

In the last step, we have rearranged the sum in increasing powers of x . now the m -sum is

$$(-1)^k \sum_{m=0}^k \binom{M+n}{m} \binom{n-m}{n-k} (-1)^m = \binom{M-1+k}{k} \quad (A-2)$$

according to a standard binomial identity,⁽¹⁷⁾ which completes the proof. The standard identity used here may be written in the form

$$\sum_{s=0}^m \binom{k}{s} \binom{k-l+m-s}{l-1} (-1)^s = (-1)^m \binom{k-l}{m} , \quad (A-3)$$

which is valid when $k > l$, and is proved by use of the obvious identity

$$(1+x)^k (1+x)^{-l} = (1+x)^{k-l} .$$

Each factor on the left is expanded in a binomial series and the resulting double sum reordered in powers of x . The desired result follows by equating to the expansion of the right side.

To prove (5-20), we need an intermediate result. Consider the identity

$$[1+x(1+y)]^m = (1+x)^m \left(1 + \frac{xy}{1+x}\right)^m , \quad (A-4)$$

and expand the left side and the right-most factor on the right side:

$$\sum_{\ell} \binom{m}{\ell} x^{\ell} (1+y)^{\ell} = (1+x)^m \sum_n \binom{m}{n} \left(\frac{xy}{1+x}\right)^n$$

$$= \sum_n \binom{m}{n} (xy)^n (1+x)^{m-n} .$$

Explicit summation limits are not used because the sums are limited automatically by the binomial coefficients, as mentioned in connection with Eq. (5-19). Now we multiply both sides by $(1+y)^s$ and expand some more. The left side becomes

$$(1+y)^s \sum_{\ell} \binom{m}{\ell} x^{\ell} (1+y)^{\ell}$$

$$= \sum_{\ell} \binom{m}{\ell} x^{\ell} \sum_k \binom{\ell+s}{k} y^k$$

$$= \sum_k \left\{ \sum_{\ell} \binom{m}{\ell} \binom{\ell+s}{k} x^{\ell} \right\} y^k ,$$

and the right side becomes

$$(1+y)^s \sum_n \binom{m}{n} x^n (1+x)^{m-n} y^n$$

$$= \sum_n \binom{m}{n} x^n (1+x)^{m-n} \sum_j \binom{s}{j} y^{n+j} .$$

we introduce the new index, $k=n+j$, and eliminate n , so that the last double sum expression becomes

$$\sum_k \sum_j \binom{m}{k-j} \binom{s}{j} x^{k-j} (1+x)^{m-k+j} y^k$$

$$= \sum_k \left\{ x^k (1+x)^{m-k} \sum_j \binom{s}{j} \binom{m}{k-j} \left(\frac{1+x}{x}\right)^j \right\} y^k .$$

Equating powers of y , we obtain the desired result:

$$\sum_l \binom{m}{l} \binom{l+s}{k} x^l = x^k (1+x)^{m-k} \sum_j \binom{s}{j} \binom{m}{k-j} \left(\frac{1+x}{x}\right)^j . \quad (A-5)$$

It is necessary only that m , s and k be non-negative, and the ranges of the sums over l and j , as limited by the binomial coefficients, will depend on the relative magnitudes of m , s and k .

Now consider (5-20):

$$\sum_l \binom{A}{l} \binom{B+l}{C} x^l = x^{C-B} (1+x)^{A-C} \sum_l \binom{B}{l} \binom{A+l}{A+B-C} x^l . \quad (A-6)$$

Using (A-5) of the left side we get

$$\sum_{\ell} \binom{A}{\ell} \binom{B+\ell}{C} x^{\ell} = x^C (1+x)^{A-C} \sum_j \binom{B}{j} \binom{A}{C-j} \left(\frac{1+x}{x}\right)^j ,$$

and applied to the right:

$$x^{C-B} (1+x)^{A-C} \sum_{\ell} \binom{B}{\ell} \binom{A+\ell}{A+B-C} x^{\ell}$$

$$= x^A \sum_j \binom{A}{j} \binom{B}{A+B-C-j} \left(\frac{1+x}{x}\right)^j$$

$$= x^A \left(\frac{1+x}{x}\right)^{A-C} \sum_j \binom{A}{A-C+j} \binom{B}{B-j} \left(\frac{1+x}{x}\right)^j .$$

In the last step we replaced j by $A-C+j$, and this completes the proof since

$$\binom{A}{A-C+j} = \binom{A}{C-j}$$

and

$$\binom{B}{B-j} = \binom{B}{j} .$$

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